Exploring Numerical Range Over Finite Fields

by

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1 Introduction

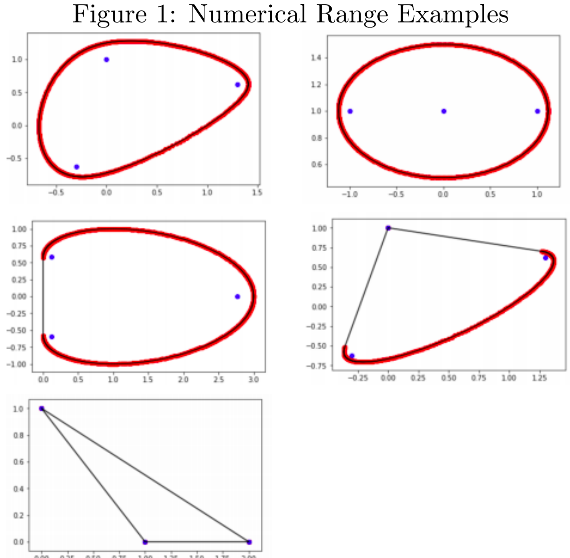
Numerical range is an interesting area of research for many mathematicians as it draws from multiple areas of mathematics, including matrix analysis, complex analysis, linear algebra, algebraic geometry, and projective geometry. Numerical range was first defined in 1918 by a German functional analyst named Otto Toeplitz [8]. The name "numerical range," however, was not used until 1932 by Marshall Stone [7].

Definition 1.1. *The* **numerical range** *of an* $n \times n$ *matrix* $A \in M_n(\mathbb{C})$ *is the set* $W(A) = \{x^*Ax : x \in \mathbb{C}^n, ||x|| = 1\}$.

The numerical range, then, is the image of a mapping from the complex unit ball in n dimensions to the complex plane with the mapping parametrized by a given matrix.

Figure 1, courtesy of Patrick X. Rault, shows 5 examples of numerical ranges. Rudolph Kippenhahn proved that these examples show all of the possible shapes that the numerical range can take [5]. The numerical range contains all of the points enclosed by the red curve as well as the red curve. The blue points in each graph are the eigenvalues of the matrix that generated that numerical range.

Figure 1: Numerical Range Examples

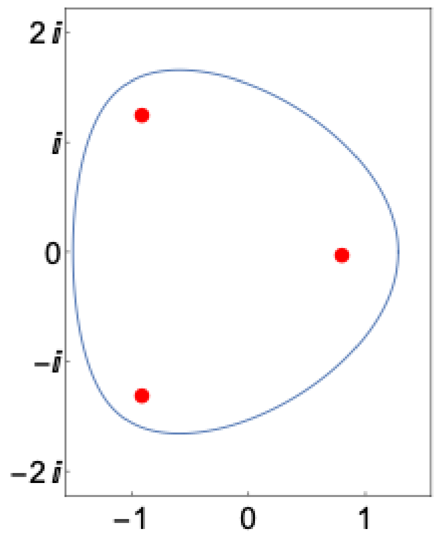


Example 1.1. *The numerical range of matrix* M_1 *is graphed in Figure 2 (courtesy of Patrick*

X. Rault). The eigenvalues of matrix M_1 are graphed with red points.

$$M_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Figure 2: Numerical Range of M_1



The numerical range is always a convex set, meaning that the line segment connecting any two points in the numerical range is completely contained in the numerical range [4]. If the matrix has complex entries, then the numerical range is also a compact set, meaning that it is both closed and bounded [4]. Additionally, the eigenvalues of the matrix are all contained in its numerical range since, for an eigenvector x from the unit ball, $x^*Ax = x^*\lambda x = \lambda x^*x = \lambda \in W(A)$.

2 Background

2.1 Complex Conjugates

To understand the classical definition of numerical range, we must understand complex numbers and their conjugates. Complex numbers have both a real part and an imaginary part. They are written as $a + bi$, where a is the real part and b is the imaginary part. The i in a complex number represents $\sqrt{-1}$. The conjugate of a complex number $a + bi$ is $a - bi$. The conjugate transpose of a matrix is obtained by transposing the matrix (reflecting it across the diagonal) and changing all of the complex entries to their conjugates. The conjugate transpose of matrix M is denoted by M^*.

Example 2.1. *For matrix* M_2*,*

$$M_2 = \begin{bmatrix} 2+6i & 1+3i \\ 0+4i & 5+0i \end{bmatrix}.$$

the conjugate transpose, M_2^**, is*

$$M_2^* = \begin{bmatrix} 2-6i & 0-4i \\ 1-3i & 5-0i \end{bmatrix}.$$

In order to obtain the numerical range, we multiply the conjugate transpose of a unit vector by the given matrix and then multiply that result by the unit vector. A **unit vector** x is a vector whose length is 1, denoted by $||x|| = \sqrt{x^*x} = 1$. The length of a vector is called its norm. Thus, a unit vector has a norm of 1. For a complex vector x, the **norm-squared** of x is x^*x.

This multiplication is written as x^*Ax in our definition of numerical range, where x is a unit vector and A is our fixed matrix. We repeat this multiplication for every unit vector, which generates the set of all points in the numerical range. Since the outputs of x^*Ax are complex, the numerical range can be graphed in the complex plane, meaning that the horizontal axis of this plane represents the real part of the numbers and the vertical axis of this plane represents the imaginary part of the numbers. A significant amount of numerical range research has been and continues to be done on the numerical range of matrices with complex numbers. There are also many variations of the classical numerical range that have been of great interest to mathematicians.

2.2 Normalization

Because the numerical range is a mapping from the complex unit ball to the complex plane, we deal quite often with unit vectors. We use a process called normalization to make any vector into a unit vector. This is done by dividing the vector by its length. The resulting vector is a unit vector pointing in the same direction as the original.

2.3 Unitary Matrices

Oftentimes when we have a particularly complicated matrix, or a matrix that is difficult to work with, it is beneficial to find a matrix that is similar to our matrix but is simpler, or easier to work with. Simpler matrices could be diagonal matrices in which all non-diagonal entries are zero. Another example of simpler matrices could be upper triangular matrices in which all entries except for those on or above the diagonal are zero. If we can find a matrix like one of these that is mathematically similar to our complicated matrix, it could be much easier to study.

We use unitary similarity to do this, as it has properties that are useful for the study of numerical range.

Definition 2.1. *A matrix U is* **unitary** *if $U^*U = I$. Two matrices A and B are* **unitarily similar** *if $U^*AU = B$ for a unitary matrix U [4].*

We use unitarily similar matrices while studying the numerical range because the unitary similarity transformation preserves the numerical range.

Theorem 2.1. *For two unitarily similar matrices A and B, $W(A) = W(B)$ [4].*

Proof. Let $A, B \in M_n(\mathbb{C})$ be unitarily similar. Then $B = U^*AU$ for some unitary U. We wish to show that $W(A) = W(B)$.

Suppose $v \in W(A)$. Then there exists a vector y with $\|y\|^2 = 1$ such that $v = y^*Ay$. Because U is unitary, $\|U(z)\|^2 = 1$ for any vector z with $\|z\|^2 = 1$. Thus there exists a vector u with $\|u\|^2 = 1$ such that $Uu = y$ so $v = y^*Ay = (Uu)^*A(Uu) = (u^*U^*)A(Uu) = u^*(U^*AU)u \in W(B)$. Thus, $W(A) \subset W(B)$.

Now suppose $w \in W(B)$. Then there exists a vector x with $\|x\|^2 = 1$ such that $w = x^*Bx = x^*(U^*AU)x = (x^*U^*)A(Ux) = (Ux)^*A(Ux)$. Since U is unitary, then $\|U(x)\|^2 = 1$. Therefore $w \in W(A)$. Thus, $W(B) \subset W(A)$.

Since $W(A) \subset W(B)$ and $W(B) \subset W(A)$, then $W(A) = W(B)$.

□

This means that if we can find a simpler matrix to which our matrix is unitarily similar, then we can study the numerical range of the simpler one and draw the same conclusions about the numerical range of the more complicated matrix.

Many numerical range characterizations consider unitarily reducible and irreducible matrices, which require the direct sum operation.

A direct sum of two matrices A and B, denoted by $A \bigoplus B$ results in a new matrix with A in the upper left corner and B in the lower right corner and zeros elsewhere. For example,

$$\begin{bmatrix} 4 & 2 & 9 \\ 6 & 1 & 8 \\ 2 & 9 & 4 \end{bmatrix} \oplus \begin{bmatrix} 4 & 5 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 9 & 0 & 0 \\ 6 & 1 & 8 & 0 & 0 \\ 2 & 9 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix}$$

.

An $n \times n$ matrix A is called **unitarily reducible** if there is a unitary matrix U such that $U^*AU = A_1 \oplus A_2$ with $A_1 \in M_k$ and $A_2 \in M_{n-k}$ for $1 < k < n$. If no such unitary matrix exists, then A is called **unitarily irreducible** [4].

2.4 Moving The Numerical Range

Another property of numerical range that is helpful to us is the fact that we can rotate and translate the numerical range to a more convenient spot on the plane. We rotate the numerical range by multiplying the matrix by a scalar. This turns the graph of the numerical range in the plane. We translate the numerical range by adding a scalar multiple of the identity to the matrix. This slides the graph of the numerical range to another place in the plane. For example, we could translate the numerical range to the first quadrant so that each element of the numerical range is nonnegative, or we could rotate it in such a way that a certain tangent line is vertical, which would make some calculations simpler.

Both rotation and translation preserve the numerical range. In other words, any rotating or translating that we do to the matrix does the same to the numerical range.

Theorem 2.2. *For all $A \in M_n(\mathbb{C})$ and $a, b \in \mathbb{C}$, $W(aA + bI) = aW(A) + b$ [4].*

Proof. Let $A \in M_n$ and $a, b \in \mathbb{C}$. Let $v \in W(aA + bI)$. Then there exists an $x \in \mathbb{C}^n$ with $\|x\|^2 = x^*x = 1$ such that

$$v = x^*(aA + bI)x = x^*aAx + x^*bIx = ax^*Ax + bx^*Ix = ax^*Ax + bx^*x = ax^*Ax + b.$$

Therefore, $v \in aW(A) + b$, so $W(aA + bI) \subset aW(A) + b$.

Now let $w \in aW(A) + b$. Then there exists an $y \in \mathbb{C}^n$ with $\|y\|^2 = y^*y = 1$ such that

$$w = ay^*Ay + b = ay^*Ay + by^*y = ay^*Ay + by^*Iy = y^*aAy + y^*bIy = y^*(aA + bI)y.$$

Therefore, $w \in W(aA + bI)$, which implies that $aW(A) + b \subset W(aA + bI)$.

Since $W(aA + bI) \subset aW(A) + b$ and $aW(A) + b \subset W(aA + bI)$, then $W(aA + bI) = aW(A) + b$.

□

2.5 Finite Fields

In this research we adapt the classical definition of numerical range by redefining it over finite fields. Therefore, we must understand what a finite field is. For our purposes, a **field** can be understood as a set for which the operations of addition and multiplication are defined and act as they do on the set of real numbers. Specifically, fields are closed under addition and multiplication, they have additive and multiplicative inverses and identities, and both addition and multiplication are associative.

One example of a field is the rational numbers because we can perform the operations of addition and multiplication on the set of rational numbers, and they act as we would expect them to.

A finite field, then, is a field whose set is finite. One example of a finite field, and the one we use most often in this research, is $\mathbb{Z}_7$. This finite field, $\mathbb{Z}_7$, is the set of integers $\{0, 1, 2, 3, 4, 5, 6\}$. Our arithmetic operations in $\mathbb{Z}_7$ are cyclical. The best way to understand how arithmetic in this field works is to imagine these numbers on a clock. In $\mathbb{Z}_7$, 8 is congruent to, or equal to, 1 because if you count 8 numbers from 0, you end up back at 1 due to its cyclical nature. This is called modular arithmetic, and we would say that "8 is congruent to 1 mod 7" and write $8 \equiv 1 \bmod 7$. Or, we could say that "4 plus 6 mod 7 is 3" and write $(4 + 6) \bmod 7 \equiv 3$.

In the general case, we use $\mathbb{Z}_p$ for a prime number p. This would be the set of integers from 0 to $p - 1$. In this research, we extend our field by adjoining a $\sqrt{-1}$ to the field of $\mathbb{Z}_p$. We call this field extension $\mathbb{Z}_p[i]$. Elements of $\mathbb{Z}_p[i]$ have a real and an imaginary part, similar to the complexes, and both the real and imaginary parts are from the set of integers mod p. This is written as $\mathbb{Z}_p[i] = \{x + yi \colon x, y \in \mathbb{Z}_p\}$. For example, $2 + 6i$, $4 + 0i$, and $5 + 1i$ are in $\mathbb{Z}_7[i]$.

When we add numbers in $\mathbb{Z}_p[i]$, we add the real parts mod p and the imaginary parts mod p. For example, $(3 + 4i) + (5 + 2i) = 8 + 6i \equiv (1 + 6i) \bmod 7$. When we multiply numbers in $\mathbb{Z}_p[i]$, we use the distributive property mod p. For example, $(5 + 1i)(2 + 6i) = (10 + 30i + 2i + 6i^2) = (10 + 32i + 6(-1)) = (10 + 32i - 6) = (4 + 32i) \equiv (4 + 4i) \bmod 7$.

Since $-1 \notin \mathbb{Z}_p$, then in the case of the $\mathbb{Z}_p[i]$ field, $i = \sqrt{p - 1}$ because $-1 \equiv (p - 1) \bmod p$. To make $\mathbb{Z}_p[i]$ a proper field extension, i cannot be in $\mathbb{Z}_p$. Thus, we make $p \equiv 3 \bmod 4$. This guarantees that $\sqrt{p - 1}$ is not in $\mathbb{Z}_p$.

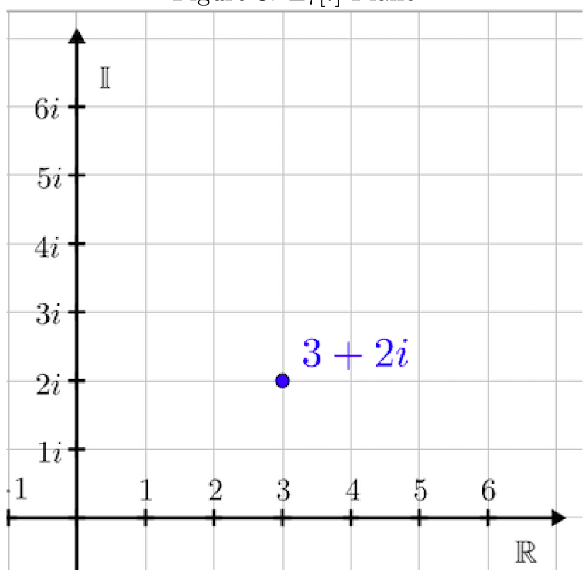
We can graph the elements of $\mathbb{Z}_p[i]$ just like we can with the elements of $\mathbb{C}$. Since $\mathbb{Z}_p[i]$ is a finite field, however, its plane has finitely many points. These points are the points on the grid. Figure 3 shows $\mathbb{Z}_7[i]$, with $3 + 2i$ graphed on it.

For an element $x = a + bi$ in $\mathbb{Z}_p[i]$, the **norm-squared** is given by $(a^2 + b^2) \bmod p$. For example, the norm-squared of $3 + 4i$ is $\|3 + 4i\|^2 = (3^2 + 4^2) \bmod p = (9 + 16) \bmod p = 25 \bmod p = 4$.

A **circle** with radius r and center c in $\mathbb{Z}_p[i]$ is defined as the set of points that is a distance of r from c. In this field, distance is given by the norm-squared of the difference between the points.

For this research, we analyzed matrices whose entries are in $\mathbb{Z}_p[i]$.

Figure 3: $\mathbb{Z}_7[i]$-Plane



3 Numerical Range Over Finite Fields

The numerical range of a matrix over a finite field was defined by Coons et al. in 2016 [3].

Definition 3.1. *Let p be a prime congruent to $3 \bmod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. We define $W(M)$, the* **finite field numerical range** *of M, to be $W(M) = \{x^*Mx : x \in \mathbb{Z}_p[i]^n, x^*x = 1\}$.*

Coons et al. also classified the numerical range of certain 2×2 matrices in the same paper [3].

Theorem 3.1 (Theorem 1.2 in Coons et al.)**.** *Let p be a prime congruent to $3 \bmod 4$ and let $M \in M_2(\mathbb{Z}_p[i])$. Assume further that all eigenvalues of M are in $\mathbb{Z}_p[i]$ and that the eigenvectors $v \in \mathbb{Z}_p[i]^2$ satisfy $v^*v \neq 0$. Then the numerical range $W(M)$ falls into one of the following classes of subsets of the plane $\mathbb{Z}_p[i]$. Furthermore, each class occurs for some $M \in M_2(\mathbb{Z}_p[i])$.*

(a) If M is a scalar multiple of the identity matrix, then $W(M)$ is a point, so $|W(M)| = 1$.

(b) *If M is unitarily reducible, then $W(M)$ is a line so $|W(M)| = p$.*

(c) *If M is unitarily irreducible and has only one eigenvalue, then $W(M)$ is the origin together with a set of $(p-1)/2$ concentric circles, with $|W(M)| = (p^2+1)/2$.*

(d) *Otherwise, $W(M)$ is a set of circles whose centers are the set of elements on some line through the origin, two circles of which are degenerate and consisting of just one point, with $|W(M)| \leq p^2 - p$.*

Overall, we have $1 \leq |W(M)| \leq p^2 - p$, with both 1 and $p^2 - p$ attainable for some prime p.

The notation $|W(M)|$ denotes the size of the numerical range of M. This is the number of points that are in the numerical range. Thus, when we say that $|W(M)| = 49$, we are saying that the numerical range of M has 49 points.

As this theorem tells us, the numerical range over finite fields is often a set of circles. This is not always obvious, as circles in a finite field are sometimes difficult to see, but Figure 4 (courtesy of Patrick X. Rault, one of the authors of [3]) shows a graph of the numerical range over $\mathbb{Z}_7[i]$ of the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Matrix A is a unitarily irreducible matrix with only one eigenvalue. Thus, as stated in Theorem 3.1, the numerical range is the origin, which is the degenerate circle, together with a set of 3 concentric circles.

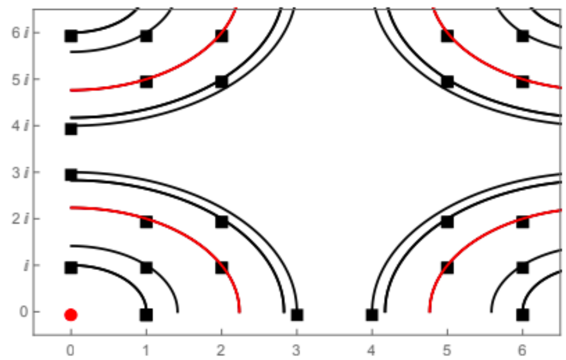
The three circles (and the origin) are drawn in the graph. One of the circles has a radius squared of 1, another has a radius squared of 2, and the third has a radius squared of 5. When these circles would go off of the plane, they begin again at another point on the plane because of the cyclical nature of $\mathbb{Z}_7[i]$. We can see the circle centered at the origin with a radius squared of 5 graphed in red. Its points are $\{1+2i, 1+5i, 2+1i, 2+6i, 5+1i, 5+6i, 6+2i, 6+5i\}$.

Coons et al. used three important results in order to prove Theorem 3.1. First, they extended the result that unitary similarity preserves the numerical range (as stated in Theorem 2.1) to the finite field case.

Lemma 3.2 (Lemma 2.6 in [3]). *Let $M, U \in M_n(\mathbb{Z}_p[i])$ with U unitary and p a prime congruent to $3 \bmod 4$. Then $W(M) = W(U^*MU)$.*

Coons et al. also extended the rotation and translation of the numerical range (Theorem 2.2) to matrices over finite fields.

Figure 4: Circles in a Numerical Range Graph



Lemma 3.3 (Lemma 2.7 in [3]). *Let p be a prime congruent to $3 \bmod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. For any $a, b \in \mathbb{Z}_p[i]$ we have $W(aM + bI) = aW(M) + b$.*

We know that every complex square matrix is unitarily similar to an upper triangular matrix. This result was extended to the case of 2×2 matrices over $\mathbb{Z}_p[i]$ [3].

Proposition 3.4 (Schur's Theorem for Finite Fields, Proposition 3.4 in [3]). *Let p be a prime congruent to $3 \bmod 4$ and let $M \in M_2(\mathbb{Z}_p[i])$. Assume further that all eigenvalues of M are in $\mathbb{Z}_p[i]$, and that the eigenvectors $v \in \mathbb{Z}_p[i]^2$ satisfy $v^*v \neq 0$. Then there exists a unitary matrix U for which U^*MU is upper triangular.*

Lemmas 3.2 and 3.3 allow us to make the lower right entry of every matrix a 0 and to scale every matrix such that the first nonzero entry is a 1. Schur's Theorem allows us to transform all 2×2 matrices into upper triangular matrices. Therefore, these three results together allow us to classify all 2×2 matrices (except for diagonal and unitarily reducible matrices, which are classified separately) into two forms, Special Form A and Special Form B:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix},$$

where $a \in \mathbb{Z}_p[i]$.

4 Numerical Ranges of 3×3 matrices in $\mathbb{Z}_7[i]$

Our goal in this research was to begin extending these results to the case of 3×3 matrices. As we investigated different matrices for this research, we wrote 8 computer programs in Python to assist in our exploration. One program, written by Aaron Monroe for Mathematics Research Seminar in Spring 2019, determines the size of the numerical range of any 2×2 matrix inputted into the code and generates a plot of the numerical range. For this research, we collaborated with Aaron Monroe to create a program that serves the same function in the 3×3 case. Two other programs handle matrices of special forms and help us to cycle through all possible matrices of these forms and analyze their numerical ranges. Two more programs help us to find the circles that the numerical ranges of 2×2 matrices generate. Finally, two other programs isolate the points in the numerical range that only appear on one circle.

In order to work toward classifying 3×3 matrices as they were classified in the 2×2 case, we needed to be sure that previous findings could be extended to the 3×3 case. Unfortunately, the proof of Schur's Theorem in the complex case requires the Gram–Schmidt process, which requires normalization. Since normalization requires division by the norm of a vector, and since the norm-squared can be 0 in the finite field case, we were not able to extend Schur's Theorem to the 3×3 case. In the 2×2 case, Coons et al. only needed to create one 0 entry in order to make the matrix upper triangular and they were able to do so by constructing a second vector based on the entries of the first. This allowed them to avoid the use of Gram–Schimidt.

As we explored ways of extending these results, we narrowed our focus to investigate the numerical ranges of matrices that fit into or were transformations of two special forms: Special Form G and Special Form H, defined below. We worked mainly in $\mathbb{Z}_7[i]$, although we are fairly confident that these results can be extended to the general $\mathbb{Z}_p[i]$ case.

4.1 Special Form G

Special Form G is the 3×3 matrix that results from a direct sum of Special Form A with a 1×1 block matrix. This means that Special Form G is generated with a matrix of form A in the upper left corner and a 1×1 matrix in the lower right corner. The rest of the entries are 0.

$$G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{bmatrix}.$$

In $\mathbb{Z}_7[i]$, there are 49 complex numbers as options for element b because there are 7 options for the real part of b and there are 7 options for the imaginary part of b. Thus, there are 49 matrices that fit Special Form G. We were able to completely classify the numerical range of matrices that fit into Special Form G in $\mathbb{Z}_7[i]$ through the method of exhaustion using our programs. In the case of all the complexes, there are infinitely many matrices that could be in this form.

Lemma 4.1. *Given a matrix $M \in M_3(\mathbb{Z}_7[i])$ of form G, if $b = \{0+3i, 0+4i, 1+i, 1+6i, 3+0i, 4+0i, 6+1i, 6+6i\}$, then $W(M)$ is the union of $\{b\}$ and the complement of a line through b. Otherwise, $W(M) = \mathbb{Z}_7[i]$.*

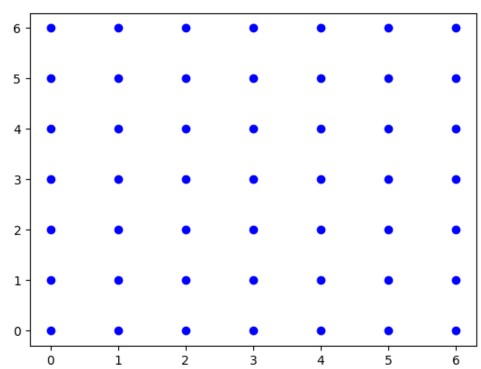
Proof. We used one of our programs to compile a list of all of the b elements that resulted in a numerical range other than all of $\mathbb{Z}_7[i]$. Then we ran the program that calculates the numerical range of any 3×3 matrix for each of these b points to analyze what the numerical range looks like in each of these cases.

□

Example 4.1. *Matrix M_5 is of form G and its numerical range is $\mathbb{Z}_7[i]$. The numerical range of M_5 is graphed in Figure 5. Notice that $3+4i$ is not in the list of b points that give a unique numerical range. Thus, $W(M_5) = \mathbb{Z}_7[i]$.*

$$M_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3+4i \end{bmatrix}.$$

Figure 5: Numerical Range of M_5

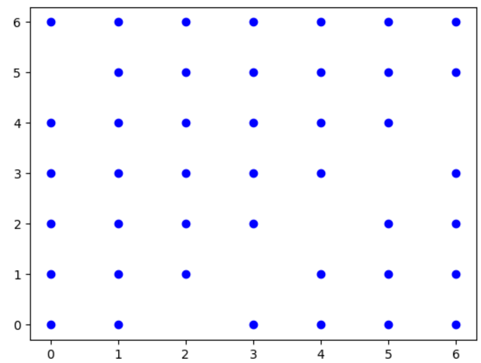


Example 4.2. *Matrix M_6 is of form G and its numerical range is $\mathbb{Z}_7[i]$ with six points on one line removed. The numerical range of M_6 is graphed in Figure 6. Notice that $1+6i$ is in the list of b points for which the size of the numerical range of the associated matrix is 43. Also notice that $1+6i$ is in the numerical range of matrix M_6, but the remaining six points on the line $y = x+5$ are*

removed.

$$M_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1+6i \end{bmatrix}.$$

Figure 6: Numerical Range of M_6



One question that still remains open about this form is why these b points in particular cause six points to be removed from $\mathbb{Z}_7[i]$ to form the numerical range. All of these b points are in the numerical range of the 2×2 A block (graphed in Figure 4), but they are only 8 out of 25 points in the numerical range of the 2×2 A block of Form G. A matrix of Form G with any of the other points in the numerical range of the 2×2 matrix inputted as b has a numerical range of $\mathbb{Z}_7[i]$. Another question for future research is why the removed line is removed. For example, why in Example 4.2 is the line $y = x + 5$ removed? Also, this lemma could be extended to $\mathbb{Z}_p[i]$, not just $\mathbb{Z}_7[i]$.

4.2 Special Form H

Special Form H is the 3×3 matrix that results from a direct sum of Special Form B with a 1×1 block matrix. This means that Special Form H is generated with a matrix of form B in the upper left corner and a 1×1 matrix in the lower right corner. The remaining entries are 0.

In many cases, several of the points on the numerical range lie on two of the circles that make up the numerical range, while some lie only on one.

$$H = \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{bmatrix}.$$

Lemma 4.2. *Given a matrix $M \in M_3(\mathbb{Z}_7[i])$ of form H,*

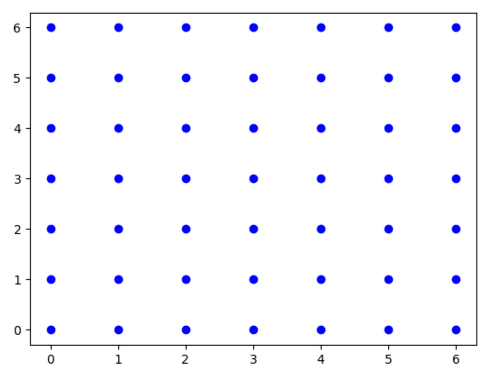
- *if a does not have norm-squared 6 and b is one of the points that lies on exactly one of the circles that make up $W(A)$, or*
- *if a is of norm-squared 6 and b has a real part of 4*

then $W(M) =$ is the union of $\{b\}$ and the complement of a line through b. Otherwise, $W(M) = \mathbb{Z}_7[i]$.

Example 4.3. *Matrix M_7 below is of form H and its numerical range is $\mathbb{Z}_7[i]$. The numerical range of M_7 is graphed in Figure 7. The points in the numerical range of the 2×2 block B matrix that lie on only one circle are $\{1+0i, 6+4i, 6+3i, 4+6i, 4+1i, 0+0i, 2+3i, 2+4i\}$. This is calculated by one of our programs finding all of the circles that make up the numerical range and noting which points are only located on one circle. Note that $2+1i$ is not in this list.*

$$M_7 = \begin{bmatrix} 1 & 3+4i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2+1i \end{bmatrix}.$$

Figure 7: Numerical Range of M_7



Example 4.4. *Matrix M_8 is of form H and its numerical range is $\mathbb{Z}_7[i]$ with six points on one line removed. The numerical range of M_8 is graphed in Figure 8.*

$$M_8 = \begin{bmatrix} 1 & 5+2i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4+3i \end{bmatrix}.$$

Figure 8: Numerical Range of M_8

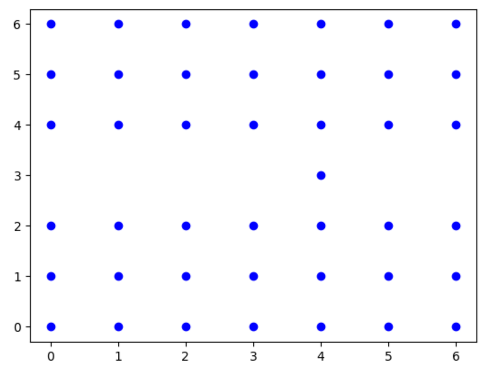
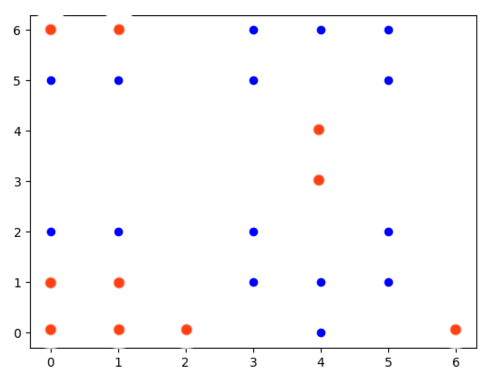


Figure 9 shows the numerical range of the B block of matrix M_8 with the points that lie on only one circle in red.

Figure 9: Numerical Range of block B of M_8



The points in the numerical range of the B block that lie on only one circle are $\{0+1i, 0+6i, 2+0i, 0+0i, 1+0i, 4+3i, 4+4i, 6+0i, 1+1i, 1+6i\}$. Notice that $4+3i$ lies on only one circle of the numerical range of the B block. Also notice that $4+3i$ is in the numerical range of matrix M_8, but the remaining points on the line $y=3$ are removed.

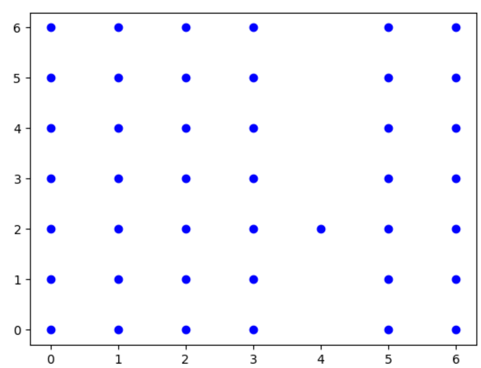
In attempting to prove this, we used one of our programs to run through the numerical range of all of the matrices of Form H. When a matrix was found to have 43 points in the numerical range instead of 49, our program then recorded the a and b entries of that matrix. Additionally, we used another of our programs to generate a list of the points that lie on only one circle of the numerical range for every 2×2 block B matrix. Then, for every a, we compared the list of b's that together with a formed a matrix with a numerical range size of 43 to the list of the points on only one circle of the B matrix with that given a. These lists matched for every matrix, except when the norm squared of a was 6.

When the norm squared of a is 6, the list of points that lie on only one circle is substantially larger than the list of points that lie on only one circle when the norm squared of a is not 6. Additionally, the line missing from a matrix of Form H with the norm squared of a being 6 is always the vertical line $x = 4$. Why these occur are questions for future research.

Example 4.5. *Matrix M_9 is of Form H with the norm squared of a being 6 and b with a real part of 4.The numerical range of M_9 is graphed in Figure 10.*

$$M_9 = \begin{bmatrix} 1 & 3+5i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4+2i \end{bmatrix}.$$

Figure 10: Numerical Range of M_9



5 Conclusion

Classifying the numerical ranges over a finite field is challenging. Because of the restrictions of a finite world, extending the general classification achieved by Kippenhahn for 3×3 matrices over the complexes has proven to be quite difficult. It took over 30 years for a full classification of numerical ranges over the complexes in the 3×3 case to appear after classical numerical ranges were introduced to the mathematical community. In addition, the extension of this classification to higher dimensions remains an open problem. Some classifications have been made of certain matrices up to the 4×4 case, but a general classification is still an unrealized goal.

We have focused on a particular class of 3×3 matrices in $\mathbb{Z}_7[i]$ and were able to classify the numerical ranges of two forms which have produced interesting results and even more interesting questions. The opportunities for future research here are countless.

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