Coadjoint Orbits for the Unipotent Group of Upper Triangular Matrices

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1. Introduction

The subject of this paper is coadjoint orbits for the unipotent group of $n \times n$ upper triangular matrices, the classification of which remains an unsolved problem with only limited partial progress [3, 4, 5, 6, 8].

The idea of an orbit in mathematics can be demonstrated through the example of rotations about the origin. Within the Cartesian plane we can select any point and rotate it about the origin a certain number of degrees. For instance, if we select the point $(1, 0)$ and rotate about the origin 90° counterclockwise we obtain the point $(0, 1)$, as illustrated in Figure 1.

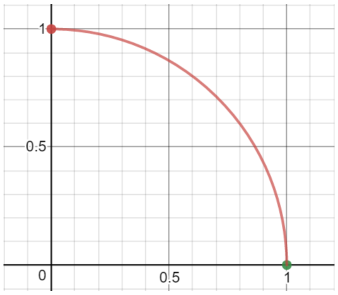

Figure 1. 90 degree rotation

While a specific rotation takes our initial point $(0, 1)$ to one new point, we can consider all possible rotations about the origin. Doing this produces a circle centered at the origin with radius 1. This circle is called the orbit of $(0, 1)$. We define the **orbit** of a point under the action of rotation about the origin to be the set of all points that can result from such rotations. In almost all cases this results in a circle. However, there is one point that is never moved no matter what degree rotation is applied to it: the origin. In the case of the origin, any degree rotation

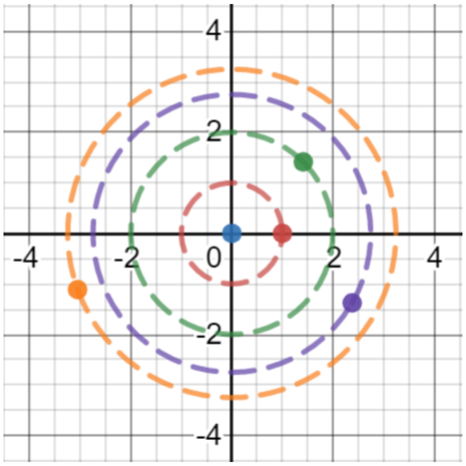

Figure 2. Orbits for various points

results in the origin. Thus the orbit of the origin is simply the set containing only the origin. Figure 2 shows the orbits of several different points.

In order to study these orbits we look for ways to classify them. Since each orbit can be generated from a single point we choose one point in each orbit to be representative of the whole. We choose the simplest point in the orbit to be this representative, where in this case simplest is defined to be points on the non-negative x-axis, as this makes this problem easier to manage. We call these representatives **basepoints**. In this case, we choose basepoints to be points on the non-negative x-axis because they have y-coordinates of 0 and no negative values. There is not necessarily a canonical way for defining basepoints for all problems, but this is clearly a logical choice as these points have a maximal number of zeros. Under this system we can select the non-negative x-axis as our set of basepoints for all orbits. In doing so, we have taken each orbit, a set of infinitely many points, and classified it in terms of a single basepoint.

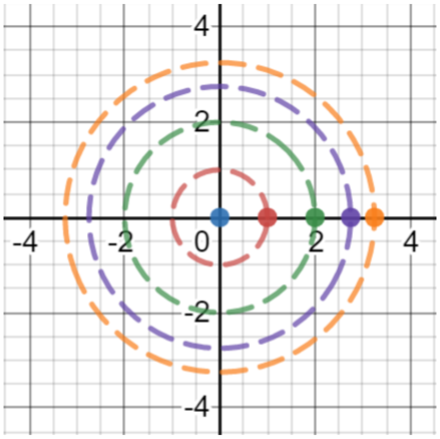

Figure 3. Orbits with their basepoints

In the rotations example we select points in the Cartesian plane and act on them through rotations about the origin. Note that the set of rotations about the origin is group. The subject of this thesis is coadjoint orbits for the unipotent group of $n \times n$ matrices. This topic is similar to orbits resulting from rotations about the origin, however with a different group acting on a different space. In the rotations example the space being acted on is the set of points in the Cartesian plane. For coadjoint orbits the space being acted on is $\mathfrak{l}_n$, the set of lower triangular $n \times n$ matrices with 0's along the diagonal. In the rotations example the group that is acting on the space is the set of rotations about the origin. For coadjoints orbits the group acting on the space is G_n, the set of $n \times n$ upper triangular matrices with 1's along the diagonal.

The group G_n acts on the space $\mathfrak{l}_n$ via the **coadjoint action**. For $g \in G_n$ and $Y \in \mathfrak{l}_n$ the coadjoint action of g on Y, denoted $\mathrm{Ad}^*(g)Y$, is defined as the lower triangularization of $g^{-1}Yg$. As in the rotations example, where an orbit is the set of all possible new points that could result from the group action, the **coadjoint**

orbit of a matrix Y is the set of all possible new matrices the coadjoint action could produce. In other words, for Y in $\mathfrak{l}_n$, the coadjoint orbit of Y is equal to

$$\{\mathrm{Ad}^*(g)Y : g \in G_n\}.$$

In this thesis we give results that aid in classifying coadjoint orbits for any $n \times n$ matrix Y by finding basepoints. As in the rotations example where basepoints are points with the maximum number of zeros, our goal is to develop a theory that will allow for a similar classification for coadjoint orbits.

In Section 2, we review linear algebra background for this problem. Most of this is well-known, and served as the starting point for our work. In Section 3, we give an explicit formula for the coadjoint action that, to our knowledge, has not yet appeared in the literature. In Section 4, we develop a general theory for the classification of coadjoint orbits, culminating in the Decomposition Theorem 4.3, which establishes a correspondence between orbits and basepoints. In Section 5, we present an example for how this classification theory is applied. The reader is encouraged to refer to section 5 while reading section 4 to gain greater intuition regarding the various definitions given.

2. Linear Algebra Background

Let F be a field. We consider $n \times n$ matrices with entries in F. For X, an $n \times n$ matrix, let $X_{i,j}$ denote the (i, j) entry of X. Let $\mathfrak{g}_n$ be the set of $n \times n$ upper triangular matrices with 0's on the diagonal. Note that $\mathfrak{g}_n$ is a vector space over F since it is closed under matrix addition and scalar multiplication. Let 0_n denote the $n \times n$ zero matrix. Let $e_{i,j}$ be the $n \times n$ matrix where all entries are 0 except for the (i, j) entry which is equal to 1. Note that these $e_{i,j}$ for $1 \leq i < j \leq n$ form a basis for $\mathfrak{g}_n$ since for each $X \in \mathfrak{g}_n$, there exists unique $X_{i,j} \in F$ such that

$$X = \sum_{1 \leq i < j \leq n} X_{i,j} e_{i,j}.$$

Recall that for V and W vector spaces over F, the map $T : V \to W$ is a **linear transformation** if

$$T(v_1 + v_2) = T(v_1) + T(v_2) \text{ and } T(cv_1) = cT(v_1)$$

for any $v_1, v_2 \in V$ and $c \in F$. If $W = F$, then T is called a **linear functional** on V. Let $\mathfrak{g}_n^*$ be the set of linear functionals on $\mathfrak{g}_n$. Note that for $f_1, f_2 \in \mathfrak{g}_n^*$, we have $f_1 + f_2 \in \mathfrak{g}_n^*$, and for $c \in F$, we have $cf_1 \in \mathfrak{g}_n^*$. Thus, $\mathfrak{g}_n^*$ is also a vector space over F.

Let $\mathfrak{l}_n$ be the set of $n \times n$ lower triangular matrices with 0's on the diagonal. We will show that $\mathfrak{l}_n$ is isomorphic to $\mathfrak{g}_n^*$. To do so, let $\langle \cdot, \cdot \rangle : \mathfrak{l}_n \times \mathfrak{g}_n \to F$ where $\langle Y, X \rangle = \operatorname{tr}(YX)$. The pairing $\langle \cdot, \cdot \rangle$ is called the **trace form**. Recall that for any $n \times n$ matrices A and B, and any $c \in F$, we have

(i) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$,

(ii) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$,

(iii) $\operatorname{tr}(cA) = c \operatorname{tr}(A)$.

For $Y \in \mathfrak{l}_n$, let $f_Y : \mathfrak{g}_n \to F$ such that $f_Y(X) = \langle Y, X \rangle$.

Lemma 2.1. *For $Y \in \mathfrak{l}_n$, $f_Y \in \mathfrak{g}_n^*$.*

Proof. Let $Y \in \mathfrak{l}_n$. Let $X_1, X_2 \in \mathfrak{g}_n$ and $c \in F$. Thus,

$$f_Y(X_1 + X_2) = \langle Y, X_1 + X_2 \rangle = \operatorname{tr}(Y(X_1 + X_2)) = \operatorname{tr}(YX_1 + YX_2)$$

$$= \operatorname{tr}(YX_1) + \operatorname{tr}(YX_2) = \langle Y, X_1 \rangle + \langle Y, X_2 \rangle = f_Y(X_1) + f_Y(X_2).$$

Note also that

$$f_Y(cX_1) = \langle Y, cX_1\rangle = \mathrm{tr}(YcX_1)$$

$$= \mathrm{tr}(cYX_1) = c\,\mathrm{tr}(YX_1) = c\langle Y, X_1\rangle = cf_Y(X_1).$$

Therefore, f_Y is a linear functional. Thus $f_Y \in \mathfrak{g}_n^*$. □

Lemma 2.2. *Let $f \in \mathfrak{g}_n^*$. Then there exists a unique $Y \in \mathfrak{l}_n$ such that $f = f_Y$.*

Proof. Let $f \in \mathfrak{g}_n^*$ and $X \in \mathfrak{g}_n$. Then

$$f(X) = f\left(\sum_{1\le i<j\le n} X_{i,j}e_{i,j}\right) = \sum_{1\le i<j\le n} X_{i,j}f(e_{i,j})$$

since f is a linear functional. Let $Y \in \mathfrak{l}_n$ such that $Y_{j,i} = f(e_{i,j})$ for all i, j where $1 \le i < j \le n$. Thus,

$$\langle Y, X\rangle = \left\langle Y, \sum_{1\le i<j\le n} X_{i,j}e_{i,j}\right\rangle = \mathrm{tr}\left(Y\sum_{1\le i<j\le n} X_{i,j}e_{i,j}\right)$$

$$= \mathrm{tr}\left(\sum_{1\le i<j\le n} X_{i,j}Ye_{i,j}\right) = \sum_{1\le i<j\le n} X_{i,j}\mathrm{tr}(Ye_{i,j})$$

$$= \sum_{1\le i<j\le n} X_{i,j}\langle Y, e_{i,j}\rangle.$$

Note that $Y = \sum_{1\le \ell<k\le n} Y_{k,\ell}e_{k,\ell}$. Thus, $Ye_{i,j} = \sum_{1\le \ell<k\le n} Y_{k,\ell}e_{k,\ell}e_{i,j}$. Since

$$e_{k,\ell}e_{i,j} = \begin{cases} e_{k,j} & \ell = i \\ 0_n & \text{otherwise,} \end{cases}$$

we have

$$Y_{k,\ell}e_{k,\ell}e_{i,j} = \begin{cases} Y_{k,i}e_{k,j} & \ell = i \\ 0_n & \text{otherwise.} \end{cases}$$

Thus

$$Ye_{i,j} = \sum_{i<k\leq n} Y_{k,i}e_{k,j}.$$

Since

$$\operatorname{tr}(e_{k,j}) = \begin{cases} 0 & k \neq j \\ 1 & k = j, \end{cases}$$

we have

$$\operatorname{tr}(Ye_{i,j}) = \operatorname{tr}\left(\sum_{i<k\leq n} Y_{k,i}e_{k,j}\right) = \operatorname{tr}(Y_{j,i}e_{j,j}) = Y_{j,i}.$$

Therefore,

$$\begin{aligned} f_Y(X) = \langle Y, X\rangle = \sum_{1\leq i<j\leq n} X_{i,j}\langle Y, e_{i,j}\rangle = \sum_{1\leq i<j\leq n} X_{i,j}Y_{j,i} \\ = \sum_{1\leq i<j\leq n} X_{i,j}f(e_{i,j}) = f(X). \end{aligned}$$

Note that Y is unique since its entries are determined by f. $\square$

Proposition 2.3. *The function $\psi : \mathfrak{l}_n \to \mathfrak{g}_n^*$ where $\psi(Y) = f_Y$ is a vector space isomorphism over F.*

Proof. Let $\psi : \mathfrak{l}_n \to \mathfrak{g}_n^*$ where $\psi(Y) = f_Y$. Lemma 1.1 and Lemma 1.2 establish that ψ is a one-to-one correspondence between $\mathfrak{l}_n$ and $\mathfrak{g}_n^*$. Let $Y_1, Y_2 \in \mathfrak{l}_n$. Then $\psi(Y_1 + Y_2) = f_{Y_1+Y_2}$. Let $X \in \mathfrak{g}_n$. Then,

$$\begin{aligned} f_{Y_1+Y_2}(X) &= \langle Y_1 + Y_2, X\rangle = \operatorname{tr}((Y_1 + Y_2)X) \\ &= \operatorname{tr}(Y_1X) + \operatorname{tr}(Y_2X) = \langle Y_1, X\rangle + \langle Y_2, X\rangle \\ &= f_{Y_1}(X) + f_{Y_2}(X). \end{aligned}$$

Hence,

$$\psi(Y_1 + Y_2) = f_{Y_1+Y_2} = f_{Y_1} + f_{Y_2} = \psi(Y_1) + \psi(Y_2).$$

Let $c \in F$. Note that $\psi(cY_1) = f_{cY_1}$. Let $X \in \mathfrak{g}_n$. Then,

$$f_{cY_1}(X) = \langle cY_1, X \rangle = \operatorname{tr}(cY_1X) = c\operatorname{tr}(Y_1X)$$

$$= c\langle Y_1, X \rangle = cf_{Y_1}(X).$$

Hence,

$$\psi(cY_1) = f_{cY_1} = cf_{Y_1} = c\psi(Y_1).$$

Therefore $\psi : \mathfrak{l}_n \to \mathfrak{g}_n^*$ is a vector space isomorphism over F. □

Let G_n be the set of all $n \times n$ upper triangular matrices with 1's on the diagonal. Note that G_n is a group since

(i) $gh \in G_n$ for all $g, h \in G_n$,

(ii) the $n \times n$ identity matrix, $I_n \in G_n$,

(iii) for any $g \in G_n, g^{-1} \in G_n$,

(iv) multiplication in G_n is associative (since matrix multiplication is associative).

We will prove part (iii) in Section 3. A **group action** of a group G on a set A is a map from $G \times A$ to A (written as $g \cdot a$, for all $g \in G$ and $a \in A$) satisfying the following properties:

(1) $g_1 \cdot (g_2 \cdot a) = (g_1g_2) \cdot a$, for all $g_1, g_2 \in G$, $a \in A$, and

(2) $e \cdot a = a$ for all $a \in A$ (where e is the identity for G).

For $a, b \in A$ we say that $a \sim b$ if and only if $a = gb$ for some $g \in G$. We can show that $\sim$ is an equivalence relation. For each $x \in A$, the equivalence class of x under $\sim$ is called the **orbit** of x under the action of G, which we denote by $[x]$. The orbits under the action of G partition the set A [2, §1.7].

For $g \in G_n$ and $X \in \mathfrak{g}_n$, let $\mathrm{Ad}(g)X = gXg^{-1}$. We know that $\mathrm{Ad}(g)X \in \mathfrak{g}_n$ and Ad is a group action when viewed as a map from $G_n \times \mathfrak{g}_n$ to $\mathfrak{g}_n$ [7]. We call Ad the **adjoint action** of G_n on $\mathfrak{g}_n$.

For $f \in \mathfrak{g}_n^*$ and $g \in G_n$, let $f_g : \mathfrak{g}_n \to F$ such that $f_g(X) = f(gXg^{-1})$ for $X \in \mathfrak{g}_n$.

Proposition 2.4. *If* $f \in \mathfrak{g}_n^*$ *and* $g \in G_n$, *then* $f_g \in \mathfrak{g}_n^*$.

Proof. Let $f \in \mathfrak{g}_n^*$ and $g \in G_n$. Let $X_1, X_2 \in \mathfrak{g}_n$ and $c \in F$. Note that

$$\begin{aligned} f_g(X_1 + X_2) &= f(g(X_1 + X_2)g^{-1}) = f((gX_1 + gX_2)g^{-1}) \\ &= f(gX_1g^{-1} + gX_2g^{-1}) \\ &= f(gX_1g^{-1}) + f(gX_2g^{-1}) = f_g(X_1) + f_g(X_2). \end{aligned}$$

Also observe that

$$f_g(cX_1) = f(g(cX_1)g^{-1}) = f(c(gX_1)g^{-1}) = cf(gX_1g^{-1}) = cf_g(X_1).$$

Thus $f_g \in \mathfrak{g}_n^*$. □

Let $f \in \mathfrak{g}_n^*$ and $g \in G_n$. Let

$$\mathrm{Ad}^*(g)f = f_g.$$

We can show that Ad^* is a group action (called the **coadjoint action**) of G_n on g_n^*. The **coadjoint orbit** of f is defined to be

$$[f] = \{\mathrm{Ad}^*(g)f : g \in G_n\}.$$

Since $\mathfrak{g}_n^*$ is isomorphic to $\mathfrak{l}_n$, the coadjoint action also acts, via the isomorphism in Proposition 2.3, on $\mathfrak{l}_n$.

Recall that as we discussed earlier, all linear functionals on $\mathfrak{g}_n$ are given by an element of $\mathfrak{l}_n$. Let $f = f_Y \in \mathfrak{g}_n^*$ and $g \in G_n$. Note that

$$\begin{aligned} f_g(X) = f(gXg^{-1}) = \langle Y, gXg^{-1}\rangle = \operatorname{tr}(Y(gXg^{-1})) \\ = \operatorname{tr}((YgX)g^{-1}) = \operatorname{tr}(g^{-1}(YgX)) = \operatorname{tr}((g^{-1}Yg)X). \end{aligned}$$

In general, $g^{-1}Yg \notin \mathfrak{l}_n$. We define the **lower projection** of A, denoted $\operatorname{proj}(A)$, and **upper projection** of a matrix A, denoted $\operatorname{proj}_U(A)$, such that

$$\operatorname{proj}(A)_{i,j} = \begin{cases} A_{i,j} & i > j \\ 0 & i \le j. \end{cases}$$

and

$$\operatorname{proj}_U(A)_{i,j} = \begin{cases} A_{i,j} & i \le j \\ 0 & i > j. \end{cases}$$

Hence, $\operatorname{proj}(g^{-1}Yg)$ is a lower triangular matrix with 0's on the diagonal, while $\operatorname{proj}_U(g^{-1}Yg)$ is an upper triangular matrix. Note that for any matrix A, $A = \operatorname{proj}(A) + \operatorname{proj}_U(A)$.

Lemma 2.5. *Let $X \in \mathfrak{g}_n$ and Z be an $n \times n$ upper triangular matrix. Then $\operatorname{tr}(ZX) = 0$.*

Proof. Let $X \in \mathfrak{g}_n$ and Z be an $n \times n$ upper triangular matrix. Hence,

$$\begin{aligned}
\operatorname{tr}(ZX) &= \sum_{k=1}^{n} (ZX)_{k,k} \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} Z_{k,l} X_{l,k} \\
&= \sum_{k=1}^{n} \left(\sum_{l=1}^{k-1} 0 \cdot X_{l,k} + \sum_{l=k}^{n} Z_{k,l} \cdot 0 \right) \\
&= 0. \qquad \square
\end{aligned}$$

Lemma 2.6. *Let $g \in G_n, Y \in \mathfrak{l}_n$ and $f = f_Y \in \mathfrak{g}_n^*$. Then $f_g = f_{\mathrm{proj}(g^{-1}Yg)}$.*

Proof. Let $g \in G_n, Y \in \mathfrak{l}_n$ and $f = f_Y \in \mathfrak{g}_n^*$. Let $X \in \mathfrak{g}_n$. Thus,

$$\begin{aligned}
f_g(X) = f_Y(gXg^{-1}) = \langle Y, gXg^{-1} \rangle = \operatorname{tr}(YgXg^{-1}) = \operatorname{tr}(g^{-1}YgX) \\
= \operatorname{tr}((\mathrm{proj}(g^{-1}Yg) + \mathrm{proj}_U(g^{-1}Yg))X) \\
= \operatorname{tr}(\mathrm{proj}(g^{-1}Yg)X) + \operatorname{tr}(\mathrm{proj}_U(g^{-1}Yg)X).
\end{aligned}$$

Note that $\mathrm{proj}_U(g^{-1}Yg)$ is an $n \times n$ upper triangular matrix. Thus, by Lemma 1.5, $\operatorname{tr}(\mathrm{proj}_U(g^{-1}Yg)X) = 0$. Therefore,

$$\begin{aligned}
f_g(X) &= \langle Y, gXg^{-1} \rangle = \operatorname{tr}(\mathrm{proj}(g^{-1}Yg)X) \\
&= \langle \mathrm{proj}(g^{-1}Yg), X \rangle = f_{\mathrm{proj}(g^{-1}Yg)}(X). \qquad \square
\end{aligned}$$

For $g \in G_n$ and $Y \in \mathfrak{l}_n$, let

$$\mathrm{Ad}^*(g)Y = \mathrm{proj}(g^{-1}Yg). \tag{1}$$

Note that for $X \in g_n$,

$$\mathrm{Ad}^*(g)f_Y(X) = \mathrm{Ad}^*(g)\langle Y, X \rangle = \langle Y, gXg^{-1} \rangle = tr(YgXg^{-1}).$$

In light of this we call $\mathrm{Ad}^*(g)f_Y = f_{\mathrm{Ad}^*(g)Y}$ the **coadjoint action** of g on Y. The **coadjoint orbit** of Y is

$$[Y] = \{\mathrm{Ad}^*(g)Y : g \in G_n\} = \{\mathrm{proj}(g^{-1}Yg) : g \in G_n\}.$$

3. Equation for $\mathrm{Ad}^*(g)Y$

In Section 3 we prove several equations, including a general expression for g^{-1} for $g \in G_n$ for $n \in \mathbb{N}$. These preliminary findings are used to establish the culmination of this section, Lemma 3.3. Recall that for $g \in G_n$ we have $g_{i,j} = 0$ when $i > j$ and $g_{i,j} = 1$ when $i = j$. Likewise for $Y \in \mathfrak{l}_n$ we have $Y_{i,j} = 0$ when $i \leq j$. We use these facts throughout this section to simplify expressions. In addition, in this section we utilize changes in the order of summation to prove Lemma 3.1 and Lemma 3.3. The first such order of summation change is found in the following paragraph.

For a double sequence $a_{\ell,k}$,

$$\begin{aligned}
\sum_{k=i}^{m} \sum_{\ell=0}^{m-k} a_{\ell,k} &= \sum_{\ell=0}^{m-i} a_{\ell,i} + \sum_{l=0}^{m-(i+1)} a_{\ell,i+1} + \ldots + \sum_{\ell=0}^{0} a_{\ell,m} \\
&= \sum_{k=i}^{m} a_{0,k} + \sum_{k=i}^{m-1} a_{1,k} + \ldots + \sum_{k=i}^{i+1} a_{m-i-1,k} + \sum_{k=i}^{i} a_{m-i,k} \\
&= \sum_{l=0}^{m-i} \sum_{k=i}^{m-\ell} a_{\ell,k}.
\end{aligned}$$

Thus

$$\sum_{k=i}^{m} \sum_{\ell=0}^{m-k} a_{\ell,k} = \sum_{l=0}^{m-i} \sum_{k=i}^{m-\ell} a_{\ell,k}. \tag{2}$$

This order of summation change will be used in the proof of the following lemma.

Lemma 3.1. *Let $g \in G_n$. For $i < j$ and $0 \le \ell \le j-i-1$, let*

$$s_{i,j,\ell} = \begin{cases} g_{i,j} & \text{if } \ell = 0, \\ \displaystyle\sum_{i<k_1<k_2<\cdots<k_\ell<j} g_{i,k_1} g_{k_1,k_2} \dots g_{k_\ell,j} & \text{if } \ell > 0. \end{cases} \tag{3}$$

Let $h \in G_n$ such that

$$h_{i,j} = \sum_{\ell=0}^{j-i-1} (-1)^{\ell+1} s_{i,j,\ell}. \tag{4}$$

Then $h = g^{-1}$.

Proof. Since G_n is a group then it is closed under matrix multiplication, so certainly $(gh)_{i,j} = 0$ for all $i > j$ and $(gh)_{i,i} = 1$ for all i. If $i < j$ then

$$\begin{aligned} (gh)_{i,j} &= \sum_{k=1}^{n} g_{i,k} h_{k,j} = \sum_{k=1}^{i-1} 0 \cdot h_{k,j} + \sum_{k=i}^{j} g_{i,k} h_{k,j} + \sum_{k=j+1}^{n} g_{i,k} \cdot 0 \\ &= \left(\sum_{k=i}^{j-1} g_{i,k} \sum_{\ell=0}^{j-k-1} (-1)^{\ell+1} s_{k,j,\ell} \right) + g_{i,j} \\ &= s_{i,j,0} + \sum_{k=i}^{j-1} \sum_{\ell=0}^{j-k-1} (-1)^{\ell+1} g_{i,k} s_{k,j,\ell}. \end{aligned}$$

By (2),

$$\sum_{k=i}^{j-1} \sum_{\ell=0}^{j-k-1} (-1)^{\ell+1} g_{i,k} s_{k,j,l} = \sum_{\ell=0}^{j-i-1} \sum_{k=i}^{j-\ell-1} (-1)^{\ell+1} g_{i,k} s_{k,j,\ell}.$$

Thus we have

$$\sum_{k=i}^{j-1} \sum_{\ell=0}^{j-k-1} (-1)^{\ell+1} g_{i,k} s_{k,j,\ell} = \sum_{\ell=0}^{j-i-1} (-1)^{\ell+1} \sum_{k=i}^{j-\ell-1} g_{i,k} s_{k,j,\ell}$$

Therefore

$$(gh)_{i,j} = s_{i,j,0} + \sum_{\ell=0}^{j-i-1} (-1)^{\ell+1} \sum_{k=i}^{j-\ell-1} g_{i,k} s_{k,j,\ell}.$$

In the special case where $j = i + 1$, we then have that

$$(gh)_{i,i+1} = s_{i,i+1,0} + \sum_{\ell=0}^{0}(-1)^{\ell+1}\sum_{k=i}^{i-\ell} g_{i,k}s_{k,i+1,\ell} = s_{i,i+1,0} - s_{i,i+1,0} = 0.$$

If instead $j > i + 1$ then

$$\begin{aligned}
(gh)_{i,j} &= s_{i,j,0} + \sum_{\ell=0}^{j-i-1}(-1)^{\ell+1}\sum_{k=i}^{j-\ell-1} g_{i,k}s_{k,j,\ell} \\
&= s_{i,j,0} + \sum_{\ell=0}^{j-i-2}(-1)^{\ell+1}\sum_{k=i}^{j-\ell-1} g_{i,k}s_{k,j,\ell} + (-1)^{j-i}s_{i,j,j-i-1} \\
&= s_{i,j,0} + \sum_{\ell=0}^{j-i-2}(-1)^{\ell+1}\left(s_{i,j,\ell} + \sum_{k=i+1}^{j-\ell-1} g_{i,k}s_{k,j,\ell}\right) + (-1)^{j-i}s_{i,j,j-i-1}
\end{aligned}$$

By (3), $\sum_{k=i+1}^{j-\ell-1} g_{i,k}s_{k,j,\ell} = s_{i,j,\ell+1}$. Thus

$$(gh)_{i,j} = s_{i,j,0} + \sum_{\ell=0}^{j-i-2}(-1)^{\ell+1}\left(s_{i,j,\ell} + s_{i,j,\ell+1}\right) + (-1)^{j-i}s_{i,j,j-i-1} = 0.$$

Therefore $gh = I_n$, hence $h = g^{-1}$. □

For a double sequence $a_{p,\ell}$,

$$\begin{aligned}
(5)\qquad \sum_{p=i+1}^{k-1}\sum_{\ell=0}^{k-p-1} a_{p,\ell} &= \sum_{\ell=0}^{k-i-2} a_{i+1,\ell} + \sum_{\ell=0}^{k-i-3} a_{i+2,\ell} + ... + \sum_{\ell=0}^{0} a_{k=1,\ell} \\
&= \sum_{\ell=0}^{k-i-2}\sum_{p=i+1}^{k-\ell-1} a_{p,\ell}.
\end{aligned}$$

This change of summation will be used in the proof of the following lemma.

Lemma 3.2. *Let $i, j \in \mathbb{N}$ such that $1 \le j < i \le n$. Then*

$$\sum_{p=i+1}^{k} g_{i,p}(g^{-1})_{p,k} = -(g^{-1})_{i,k}.$$

Proof. Note that by Lemma 3.1,

$$\begin{aligned}\sum_{p=i+1}^{k} g_{i,p}(g^{-1})_{p,k} &= g_{i,k}(g^{-1})_{k,k} + \sum_{p=i+1}^{k-1} g_{i,p} \sum_{\ell=0}^{k-p-1} (-1)^{\ell-1} s_{p,k,\ell} \\ &= g_{i,k} + \sum_{p=i+1}^{k-1} \sum_{\ell=0}^{k-p-1} (-1)^{\ell-1} g_{i,p} s_{p,k,\ell}.\end{aligned}$$

By (5),

$$\sum_{p=i+1}^{k-1} \sum_{\ell=0}^{k-p-1} (-1)^{\ell-1} g_{i,p} s_{p,k,\ell} = \sum_{\ell=0}^{k-i-2} \sum_{p=i+1}^{k-\ell-1} (-1)^{\ell+1} g_{i,p} s_{p,k,\ell}.$$

Therefore,

$$\begin{aligned}\sum_{p=i+1}^{k} g_{i,p}(g^{-1})_{p,k} &= g_{i,k} + \sum_{p=i+1}^{k-1} \sum_{\ell=0}^{k-p-1} (-1)^{\ell-1} g_{i,p} s_{p,k,\ell} \\ &= g_{i,k} + \sum_{\ell=0}^{k-i-2} \sum_{p=i+1}^{k-\ell-1} (-1)^{\ell+1} g_{i,p} s_{p,k,\ell} \\ &= g_{i,k} + \sum_{\ell=0}^{k-i-2} (-1)^{\ell+1} \sum_{p=i+1}^{k-\ell-1} g_{i,p} s_{p,k,\ell} \\ &= g_{i,k} + \sum_{\ell=0}^{k-i-2} (-1)^{\ell+1} s_{i,k,\ell+1} \\ &= g_{i,k} + \sum_{\ell=1}^{k-i-1} (-1)^{\ell} s_{i,k,\ell} \\ &= -\sum_{\ell=0}^{k-i-1} (-1)^{\ell+1} s_{i,k,\ell} \\ &= -(g^{-1})_{i,k}.\end{aligned}$$

□

Note that for a double sequence $a_{k,p}$,

$$\begin{aligned}\sum_{p=i+1}^{n}\sum_{k=p}^{n}a_{k,p} &= \sum_{k=i+1}^{n}a_{k,i+1}+\sum_{k=i+2}^{n}a_{k,i+2}+...+a_{n,n}\\ &= a_{i+1,i+1}+\sum_{p=i+1}^{i+2}a_{i+2,p}+...+\sum_{p=i+1}^{n}a_{n,p}\\ &= \sum_{k=i+1}^{n}\sum_{p=i+1}^{k}a_{k,p}.\end{aligned}$$

Thus

$$\sum_{p=i+1}^{n}\sum_{k=p}^{n}a_{k,p}=\sum_{k=i+1}^{n}\sum_{p=i+1}^{k}a_{k,p}. \tag{6}$$

This change of summation will be used in the proof of the following lemma.

Lemma 3.3. *For* $g\in G_n$, $Y\in\mathfrak{l}$, *and* $1\le j<i<n$,

$$(\mathrm{Ad}^*(g)Y)_{i,j}=Y_{i,j}+\sum_{q=1}^{j-1}g_{q,j}Y_{i,q}-\sum_{p=i+1}^{n}g_{i,p}(\mathrm{Ad}^*(g)Y)_{p,j}$$

Proof. By (1),

$$\begin{aligned}(\mathrm{Ad}^*(g)Y)_{i,j} &= \mathrm{proj}(g^{-1}Yg)_{i,j}=(g^{-1}Yg)_{i,j}=\sum_{p=1}^{n}(g^{-1})_{i,p}(Yg)_{p,j}\\ &= \sum_{p=i}^{n}(g^{-1})_{i,p}\sum_{q=1}^{n}Y_{p,q}g_{q,j}=\sum_{p=i}^{n}(g^{-1})_{i,p}\sum_{q=1}^{j}Y_{p,q}g_{q,j}\\ &= \sum_{p=i}^{n}\sum_{q=1}^{j}(g^{-1})_{i,p}Y_{p,q}g_{q,j}.\end{aligned} \tag{7}$$

Thus

$$(\mathrm{Ad}^*(g)Y)_{i,j} = \sum_{p=i}^{n}\sum_{q=1}^{j}(g^{-1})_{i,p}Y_{p,q}g_{q,j} \tag{8}$$

$$= \sum_{q=1}^{j}(g^{-1})_{i,i}Y_{i,q}g_{q,j} + \sum_{p=i+1}^{n}\sum_{q=1}^{j}(g^{-1})_{i,p}Y_{p,q}g_{q,j}$$

$$= \sum_{q=1}^{j}Y_{i,q}g_{q,j} + \sum_{p=i+1}^{n}\sum_{q=1}^{j}(g^{-1})_{i,p}Y_{p,q}g_{q,j}$$

$$= Y_{i,j}g_{j,j} + \sum_{q=1}^{j-1}Y_{i,q}g_{q,j} + \sum_{p=i+1}^{n}\sum_{q=1}^{j}(g^{-1})_{i,p}Y_{p,q}g_{q,j}$$

$$= Y_{i,j} + \sum_{q=1}^{j-1}Y_{i,q}g_{q,j} + \sum_{p=i+1}^{n}\sum_{q=1}^{j}(g^{-1})_{i,p}Y_{p,q}g_{q,j}.$$

By (7),

$$\sum_{p=i+1}^{n} g_{i,p}(\mathrm{Ad}^*(g)Y)_{p,j} = \sum_{p=i+1}^{n} g_{i,p}\sum_{k=p}^{n}\sum_{q=1}^{j}(g^{-1})_{p,k}Y_{k,q}g_{q,j}$$

$$= \sum_{p=i+1}^{n}\sum_{q=1}^{j} g_{i,p}\sum_{k=p}^{n}(g^{-1})_{p,k}Y_{k,q}g_{q,j}$$

$$= \sum_{q=1}^{j}\sum_{p=i+1}^{n}\sum_{k=p}^{n} g_{i,p}(g^{-1})_{p,k}Y_{k,q}g_{q,j}.$$

Note that by (6),

$$\sum_{p=i+1}^{n}\sum_{k=p}^{n} g_{i,p}(g^{-1})_{p,k}Y_{k,q}g_{q,j} = \sum_{k=i+1}^{n}\sum_{p=i+1}^{k} g_{i,p}(g^{-1})_{p,k}Y_{k,q}g_{q,j}.$$

Hence

$$\sum_{p=i+1}^{n} g_{i,p}(\mathrm{Ad}^*(g)Y)_{p,j} = \sum_{q=1}^{j}\sum_{p=i+1}^{n}\sum_{k=p}^{n} g_{i,p}(g^{-1})_{p,k}Y_{k,q}g_{q,j}$$

$$= \sum_{q=1}^{j}\sum_{k=i+1}^{n}\sum_{p=i+1}^{k} g_{i,p}(g^{-1})_{p,k}Y_{k,q}g_{q,j}$$

$$= \sum_{q=1}^{j}\sum_{k=i+1}^{n} Y_{k,q}\sum_{p=i+1}^{k} g_{i,p}(g^{-1})_{p,k}g_{q,j}.$$

Thus by Lemma 3.2,

$$\begin{aligned}\sum_{p=i+1}^{n} g_{i,p}(\mathrm{Ad}^*(g)Y)_{p,j} &= \sum_{q=1}^{j}\sum_{k=i+1}^{n} Y_{k,q} \sum_{p=i+1}^{k} g_{i,p}(g^{-1})_{p,k} g_{q,j} \\ &= -\sum_{q=1}^{j}\sum_{k=i+1}^{n} Y_{k,q}(g^{-1})_{i,k} g_{q,j} \\ &= -\sum_{k=i+1}^{n}\sum_{q=1}^{j} Y_{k,q}(g^{-1})_{i,k} g_{q,j}.\end{aligned}$$

Recall from (8),

$$(\mathrm{Ad}^*(g)Y)_{i,j} = Y_{i,j} + \sum_{q=1}^{j-1} g_{q,j} Y_{i,q} + \sum_{p=i+1}^{n}\sum_{q=1}^{j} Y_{p,q}(g^{-1})_{i,p} g_{q,j}.$$

Therefore

$$(\mathrm{Ad}^*(g)Y)_{i,j} = Y_{i,j} + \sum_{q=1}^{j-1} g_{q,j} Y_{i,q} - \sum_{p=i+1}^{n} g_{i,p}(\mathrm{Ad}^*(g)Y)_{p,j}. \qquad \square$$

4. Classification Definitions and Method

The following definitions are used to formulate a technique for decomposing the space of coadjoint orbits into disjoint subsets. These subsets will be quasi-affine varieties ($\mathfrak{l}_{\Psi,\Phi}$ defined below). We then describe how to iteratively decompose these quasi-affine varieties. This happens either by identifying a polynomial for which the variety can be further partitioned into two subsets (called decomposition) or by constraining to a particular subset of the variety (called restriction). This repeats until a variety is produced which is terminal, as defined below. Each terminal variety corresponds to a set of basepoints, where a basepoint is an element of $\mathfrak{l}$ that is representative of its coadjoint orbit. Throughout this section the reader is encourage to reference Section 5, as Section 5 demonstrates how the terms defined in Section 4 are used.

Fix $n \in \mathbb{N}$ and let $\mathfrak{l} = \mathfrak{l}_n$ and $G = G_n$. Since $\mathfrak{l}$ is a vector space with basis $e_{i,j}$ for $1 \leq j < i \leq n$, the map

$$(c_{2,1}, c_{3,1}, c_{3,2}, ..., c_{n,n-1}) \mapsto c_{2,1}e_{2,1} + c_{3,1}e_{3,1} + c_{3,2}e_{3,2} + ... + c_{n,n-1}e_{n,n-1}$$

defines an affine coordinate frame for $\mathfrak{l}$ thereby giving $\mathfrak{l}$ the structure of an affine space [1]. Let $F[\mathfrak{l}]$ denote the corresponding ring of polynomials.

Let $\Psi, \Phi \subseteq F[\mathfrak{l}]$. Let

$$\mathfrak{l}_{\Psi,\Phi} = \{Y \in \mathfrak{l} : \psi(Y) \neq 0 \text{ and } \phi(Y) = 0 \text{ for all } \psi \in \Psi \text{ and } \phi \in \Phi\}.$$

Let

$$\Theta \subseteq M = \{(i,j) : 1 \leq j < i \leq n\}.$$

Let

$$\mathfrak{X}_{\Psi,\Phi,\Theta} = \{(g,Y) \in G \times \mathfrak{l}_{\Psi,\Phi} : \mathrm{Ad}^*(g)Y \in \mathfrak{l}_{\Psi,\Phi} \text{ and } (\mathrm{Ad}^*(g)Y)_{t,s} = 0 \text{ for all } (t,s) \in \Theta\}.$$

For $Y \in \mathfrak{l}_{\Psi,\Phi}$, let

$$\mathfrak{X}^{Y}_{\Psi,\Phi,\Theta} = \{g \in G : (g,Y) \in \mathfrak{X}_{\Psi,\Phi,\Theta}\}.$$

We say $f \in F[\mathfrak{l}]$ is **invariant** with respect to (Ψ, Φ, Θ) if

$$f(\mathrm{Ad}^*(g_1)Y) = f(\mathrm{Ad}^*(g_2)Y)$$

for all $g_1, g_2 \in \mathfrak{X}^{Y}_{\Psi,\Phi,\Theta}$ for all $Y \in \mathfrak{l}_{\Psi,\Phi}$. We say $h \in F[\mathfrak{l}]$ is **normal** with respect to (Ψ, Φ, Θ) if there exists $f \in F[\mathfrak{l}]$ such that

$$f(\mathrm{Ad}^*(g)Y) = h(Y)$$

for all $(g, Y) \in \mathfrak{X}_{\Psi,\Phi,\Theta}$. Note that such f must be invariant with respect to (Ψ, Φ, Θ). We say that $h \in F[\mathfrak{l}]$ is **orbital** with respect to (Ψ, Φ, Θ) if h is normal with respect to (Ψ, Φ, Θ) and if for all $Y_1, Y_2 \in \mathfrak{l}_{\Psi,\Phi}$ we have

$$h(Y_1) \neq 0 \text{ and } h(Y_2) = 0 \implies [Y_1] \cap [Y_2] = \varnothing,$$

and if there exist $Y_1, Y_2 \in \mathfrak{l}_{\Psi,\Phi}$ such that $h(Y_1) \neq 0$ and $h(Y_2) = 0$. We say that (Ψ, Φ, Θ) is **genuine** if $\mathfrak{l}_{\Psi,\Phi} \neq \varnothing$ and $\mathfrak{X}^{Y}_{\Psi,\Phi,\Theta} \neq \varnothing$ for all $Y \in \mathfrak{l}_{\Psi,\Phi}$. Note that $(\varnothing, \varnothing, \varnothing)$ is genuine.

Lemma 4.1 (Decomposition Lemma)**.** *Suppose* $\Psi, \Phi \subseteq F[\mathfrak{l}]$ *and* $\Theta \subseteq M$ *such that* (Ψ, Φ, Θ) *is genuine. Let* $h \in F[\mathfrak{l}]$ *be orbital with respect to* (Ψ, Φ, Θ)*.*

(a) If $Y \in \mathfrak{l}_{\Psi,\Phi}$ *then either* $[Y] \cap \mathfrak{l}_{\Psi\cup\{h\},\Phi} = \varnothing$ *or* $[Y] \cap \mathfrak{l}_{\Psi,\Phi\cup\{h\}} = \varnothing$ *but not both.*

(b) The tuples $(\Psi \cup \{h\}, \Phi, \Theta)$ *and* $(\Psi, \Phi \cup \{h\}, \Theta)$ *are both genuine.*

Proof. Let $Y \in \mathfrak{l}_{\Psi,\Phi}$. Since either $h(Y) \neq 0$ or $h(Y) = 0$, then $Y \in \mathfrak{l}_{\Psi\cup\{h\},\Phi}$ or $Y \in \mathfrak{l}_{\Psi,\Phi\cup\{h\}}$, but not both. Note that since the identity matrix is in G, then $Y \in [Y]$.

Suppose $Y \in \mathfrak{l}_{\Psi\cup\{h\},\Phi}$. Then $[Y] \cap \mathfrak{l}_{\Psi\cup\{h\},\Phi} \neq \varnothing$. Suppose also that $[Y] \cap \mathfrak{l}_{\Psi,\Phi\cup\{h\}} \neq \varnothing$. Hence there exists $Y_1 \in [Y] \cap \mathfrak{l}_{\Psi,\Phi\cup\{h\}}$. Thus $h(Y_1) = 0$. However, since $h(Y) \neq 0$ and h is orbital with respect to (Ψ, Φ, Θ), this implies that $[Y] \cap [Y_1] = \varnothing$, which contradicts $Y_1 \in [Y]$. Therefore if $Y \in \mathfrak{l}_{\Psi\cup\{h\},\Phi}$ then $[Y] \cap \mathfrak{l}_{\Psi\cup\{h\},\Phi} \neq \varnothing$ and $[Y] \cap \mathfrak{l}_{\Psi,\Phi\cup\{h\}} = \varnothing$.

Suppose $Y \in \mathfrak{l}_{\Psi,\Phi\cup\{h\}}$. Then $[Y] \cap \mathfrak{l}_{\Psi,\Phi\cup\{h\}} \neq \varnothing$. Suppose also that $[Y] \cap \mathfrak{l}_{\Psi\cup\{h\},\Phi} \neq \varnothing$. Hence there exists $Y_1 \in [Y] \cap \mathfrak{l}_{\Psi\cup\{h\},\Phi} \neq \varnothing$. Thus $h(Y_1) \neq 0$. However, since $h(Y) = 0$ and h is orbital with respect to (Ψ, Φ, Θ), this implies that $[Y] \cap [Y_1] = \varnothing$, which contradicts $Y_1 \in [Y]$. Therefore if $Y \in \mathfrak{l}_{\Psi,\Phi\cup\{h\}}$ then $[Y] \cap \mathfrak{l}_{\Psi,\Phi\cup\{h\}} \neq \varnothing$ and $[Y] \cap \mathfrak{l}_{\Psi\cup\{h\},\Phi} = \varnothing$. This proves part (a).

Since h is orbital with respect to (Ψ, Φ, Θ), $\mathfrak{l}_{\Psi\cup\{h\},\Phi} \neq \varnothing$. Let $Y \in \mathfrak{l}_{\Psi\cup\{h\},\Phi}$. Clearly $Y \in \mathfrak{l}_{\Psi,\Phi}$. Since (Ψ, Φ, Θ) is genuine, there exists $g \in \mathfrak{X}^{Y}_{\Psi,\Phi,\Theta}$ such that $\mathrm{Ad}^*(g)Y \in \mathfrak{l}_{\Psi,\Phi}$ and $(\mathrm{Ad}^*(g)Y)_{t,s} = 0$ for all $(t, s) \in \Theta$. Recall that coadjoint orbits partition $\mathfrak{l}$, and thus $\mathrm{Ad}^*(g)Y \in [Y]$ then $[\mathrm{Ad}^*(g)Y] = [Y]$. Therefore, since h is orbital with respect to (Ψ, Φ, Θ), either $h(Y) \neq 0$ and $h(\mathrm{Ad}^*(g)Y) \neq 0$ or $h(Y) = h(\mathrm{Ad}^*(g)Y) = 0$. Since $Y \in \mathfrak{l}_{\Psi\cup\{h\},\Phi}$, then $h(\mathrm{Ad}^*(g)Y) \neq 0$, hence $\mathrm{Ad}^*(g)Y \in \mathfrak{l}_{\Psi\cup\{h\},\Phi}$. Thus $g \in \mathfrak{X}^{Y}_{\Psi\cup\{h\},\Phi,\Theta}$. Therefore $(\Psi \cup \{h\}, \Phi, \Theta)$ is genuine.

Since h is orbital with respect to (Ψ, Φ, Θ), $\mathfrak{l}_{\Psi,\Phi\cup\{h\}} \neq \varnothing$. Let $Y \in \mathfrak{l}_{\Psi,\Phi\cup\{h\}}$. Clearly $Y \in \mathfrak{l}_{\Psi,\Phi}$. Since (Ψ, Φ, Θ) is genuine, there exists $g \in \mathfrak{X}^{Y}_{\Psi,\Phi,\Theta}$ such that $\mathrm{Ad}^*(g)Y \in \mathfrak{l}_{\Psi,\Phi}$ and $(\mathrm{Ad}^*(g)Y)_{t,s} = 0$ for all $(t, s) \in \Theta$. Since coadjoint orbits partition $\mathfrak{l}$ and $\mathrm{Ad}^*(g)Y \in [Y]$, then $[\mathrm{Ad}^*(g)Y] = [Y]$. Therefore, since h is orbital with respect to (Ψ, Φ, Θ), either $h(Y) \neq 0$ and $h(\mathrm{Ad}^*(g)Y) \neq 0$ or $h(Y) = h(\mathrm{Ad}^*(g)Y) = 0$. Since $Y \in \mathfrak{l}_{\Psi,\Phi\cup\{h\}}$, then $h(\mathrm{Ad}^*(g)Y) = 0$, hence $\mathrm{Ad}^*(g)Y \in \mathfrak{l}_{\Psi,\Phi\cup\{h\}}$. Thus $g \in \mathfrak{X}^{Y}_{\Psi,\Phi\cup\{h\},\Theta}$. Therefore $(\Psi, \Phi \cup \{h\}, \Theta)$ is genuine. □

If (Ψ, Φ, Θ) is genuine and $h \in F[\mathfrak{l}]$ is orbital with respect to (Ψ, Φ, Θ), then we call $\{(\Psi \cup \{h\}, \Phi, \Theta), (\Psi, \Phi \cup \{h\}, \Theta)\}$ a **decomposition** of (Ψ, Φ, Θ) and we call $(\Psi \cup \{h\}, \Phi, \Theta)$ and $(\Psi, \Phi \cup \{h\}, \Theta)$ the **decomposition components** of (Ψ, Φ, Θ). If (Ψ, Φ, Θ) is genuine and $\Theta \subseteq \Theta' \subseteq M$ then we say (Ψ, Φ, Θ') is a **restriction** of (Ψ, Φ, Θ) if (Ψ, Φ, Θ') is genuine. We will sometimes write $(\Psi, \Phi, \Theta) \preceq (\Psi', \Phi', \Theta')$ to indicate that (Ψ', Φ', Θ') is either a decomposition component of (Ψ, Φ, Θ) or a restriction of (Ψ, Φ, Θ).

We say that (Ψ, Φ, Θ) is **restrictable** if there exists a restriction of (Ψ, Φ, Θ), i.e. there exists $\Theta' \supsetneq \Theta$ such that (Ψ, Φ, Θ') is genuine. We say genuine (Ψ, Φ, Θ) is **terminal** if (Ψ, Φ, Θ) is not restrictable and for each $Y \in \mathfrak{l}_{\Psi,\Phi}$, there exists $Y_B \in \mathfrak{l}_{\Psi,\Phi}$ such that $\mathrm{Ad}^*(g)Y = Y_B$ for all $g \in \mathfrak{X}^{Y}_{\Psi,\Phi,\Theta}$. We say that (Ψ, Φ, Θ) is **decomposable** if it is not terminal and there exists a decomposition of (Ψ, Φ, Θ), i.e. there exists $h \in F[\mathfrak{l}]$ such that h is orbital with respect to (Ψ, Φ, Θ).

For each $m \in \mathbb{Z}_{\geq 0}$ let $\mathcal{F}_m$ be a set of genuine (Ψ, Φ, Θ) such that

(1) $\mathcal{F}_0 = \{\varnothing, \varnothing, \varnothing\}$.

(2) If $(\Psi, \Phi, \Theta) \in \mathcal{F}_{m+1}$ then there exists a unique $(\Psi', \Phi', \Theta') \in \mathcal{F}_m$ such that either

  (a) (Ψ, Φ, Θ) is a decomposition component of (Ψ', Φ', Θ') or

  (b) (Ψ, Φ, Θ) is a restriction of (Ψ', Φ', Θ')

  but not both (a) and (b).

(3) If $(\Psi, \Phi, \Theta) \in \mathcal{F}_m$ and (Ψ, Φ, Θ) is not terminal then either

  (a) (Ψ, Φ, Θ) has exactly one pair of decomposition components in $\mathcal{F}_{m+1}$ or

  (b) (Ψ, Φ, Θ) has exactly one restriction in $\mathcal{F}_{m+1}$

  but not both (a) and (b).

(4) There exists $M \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{F}_m = \varnothing$ for all $m \geq M$.

We say that such $\mathcal{F}$ is a **classifier**.

Lemma 4.2. *Let $Y \in \mathfrak{l}$. Suppose $\mathcal{F}$ is a classifier. Let*

$$\mathcal{O}_m = \mathcal{F}_m \cup \{(\Psi, \Phi, \Theta) \in \mathcal{F}_k : (\Psi, \Phi, \Theta) \textit{ is terminal and } k < m\}.$$

Then for each $m \in \mathbb{Z}_{\geq 0}$*, there exists a unique* $(\Psi, \Phi, \Theta) \in \mathcal{O}_m$ *such that* $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$*.*

Proof. We do a proof by induction. Let $Y \in \mathfrak{l}$. Note that $\mathcal{O}_0 = \mathcal{F}_0 = \{(\varnothing, \varnothing, \varnothing)\}$. Since $\mathfrak{l}_{\varnothing,\varnothing} = \mathfrak{l}$, then clearly $(\varnothing, \varnothing, \varnothing)$ is unique in $\mathcal{O}_0$ such that $[Y] \cap \mathfrak{l}_{\varnothing,\varnothing} \neq \varnothing$.

Let $m \in \mathbb{Z}_{\geq 0}$. Suppose there exists unique $(\Psi, \Phi, \Theta) \in \mathcal{O}_m$ such that $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$. We have three cases: (Ψ, Φ, Θ) is either terminal, restrictable, or decomposable. First suppose (Ψ, Φ, Θ) is terminal. Then $(\Psi, \Phi, \Theta) \in \mathcal{O}_{m+1}$ and $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$. Suppose instead that (Ψ, Φ, Θ) is decomposable. Then by the Decomposition Lemma, there exists $(\Psi', \Phi', \Theta') \in \mathcal{O}_{m+1}$ a decomposition component of (Ψ, Φ, Θ) such that $[Y] \cap \mathfrak{l}_{\Psi',\Phi'} \neq \varnothing$. Finally suppose that (Ψ, Φ, Θ) is restrictable. Then there exists $(\Psi, \Phi, \Theta') \in \mathcal{O}_{m+1}$ such that $\Theta' \supsetneq \Theta$ and $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$. Thus for each $Y \in \mathfrak{l}$, there exists $(\Psi, \Phi, \Theta) \in \mathcal{O}_{m+1}$ such that $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$.

Next we prove uniqueness. Let $Y \in \mathfrak{l}$. Suppose $(\Psi_1, \Phi_1, \Theta_1), (\Psi_2, \Phi_2, \Theta_2) \in \mathcal{O}_{m+1}$ such that $[Y] \cap \mathfrak{l}_{\Psi_1,\Phi_1} \neq \varnothing$ and $[Y] \cap \mathfrak{l}_{\Psi_2,\Phi_2} \neq \varnothing$. Since $\mathcal{F}$ is a classifier then there exists $(\Psi_1', \Phi_1', \Theta_1'), (\Psi_2', \Phi_2', \Theta_2') \in \mathcal{O}_m$ such that $(\Psi_1', \Phi_1', \Theta_1') \preceq (\Psi_1, \Phi_1, \Theta_1)$ and $(\Psi_2', \Phi_2', \Theta_2') \preceq (\Psi_2, \Phi_2, \Theta_2)$. Since $\mathfrak{l}_{\Psi_1,\Phi_1} \subseteq \mathfrak{l}_{\Psi_1',\Phi_1'}$ and $\mathfrak{l}_{\Psi_2,\Phi_2} \subseteq \mathfrak{l}_{\Psi_2',\Phi_2'}$, then $[Y] \cap \mathfrak{l}_{\Psi_1',\Phi_1'} \neq \varnothing$ and $[Y] \cap \mathfrak{l}_{\Psi_2',\Phi_2'} \neq \varnothing$. Thus since $(\Psi_1', \Phi_1', \Theta_1'), (\Psi_2', \Phi_2', \Theta_2') \in \mathcal{O}_m$, then by our induction hypothesis $(\Psi_1', \Phi_1', \Theta_1') = (\Psi_2', \Phi_2', \Theta_2')$.

Since $\mathcal{F}$ is a classifier and $(\Psi_1, \Phi_1, \Theta_1) \preceq (\Psi_1', \Phi_1', \Theta_1')$ and $(\Psi_2, \Phi_2, \Theta_2) \preceq (\Psi_1', \Phi_1', \Theta_1')$ then $(\Psi_1, \Phi_1, \Theta_1)$ and $(\Psi_2, \Phi_2, \Theta_2)$ are decomposition components of $(\Psi_1', \Phi_1', \Theta_1')$. Thus by the Decomposition Lemma, either $[Y] \cap \mathfrak{l}_{\Psi_1,\Phi_1} = \varnothing$ or $[Y] \cap \mathfrak{l}_{\Psi_2,\Phi_2} = \varnothing$, which is a contradiction. Hence $(\Psi, \Phi, \Theta) \in \mathcal{O}_{m+1}$ such that $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$ is unique.

Therefore, by induction, for all $m \in \mathbb{Z}_{\geq 0}$, there exists unique $(\Psi, \Phi, \Theta) \in \mathcal{O}_m$ such that $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$. $\square$

Theorem 4.3 (Decomposition Theorem)**.** *Let $\mathcal{F}$ be a classifier. Let $\mathcal{T} = \{(\Psi, \Phi, \Theta) \in \mathcal{F}_k : k \in \mathbb{Z}_{\geq 0}, (\Psi, \Phi, \Theta)$ is terminal$\}$. Let $\mathcal{B}(\Psi, \Phi, \Theta) = \{Y_B : Y \in \mathfrak{l}_{\Psi,\Phi}\}$. Let $\mathcal{B} = \displaystyle\bigcup_{(\Psi,\Phi,\Theta)\in\mathcal{T}} \mathcal{B}(\Psi, \Phi, \Theta)$. Then there is a one-to-one correspondence between $\mathcal{B}$ and the set of coadjoint orbits of G.*

Proof. Let $Y \in \mathfrak{l}$. Let $m \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{O}_m = \mathcal{T}$. Then by Lemma 4.2 there exists unique $(\Psi, \Phi, \Theta) \in \mathcal{T}$ such that $[Y] \cap \mathfrak{l}_{\Psi,\Phi} \neq \varnothing$. Let $Y_1, Y_2 \in [Y] \cap \mathfrak{l}_{\Psi,\Phi}$. Then since (Ψ, Φ, Θ) is terminal, there exists $(Y_1)_B, (Y_2)_B \in \mathfrak{l}_{\Psi,\Phi}$ such that $\mathrm{Ad}^*(g)Y_1 = (Y_1)_B$ for all $g \in \mathfrak{X}^{Y_1}_{\Psi,\Phi,\Theta}$ and $\mathrm{Ad}^*(g)Y_2 = (Y_2)_B$ for all $g \in \mathfrak{X}^{Y_2}_{\Psi,\Phi,\Theta}$. Note that $[(Y_1)_B] = [Y_1] = [Y] = [Y_2] = [(Y_2)_B]$. Therefore there exists $g \in G$ such that $\mathrm{Ad}^*(g)(Y_1)_B = (Y_2)_B$. Note that $g \in \mathfrak{X}^{(Y_1)_B}_{\Psi,\Phi,\Theta}$. Note also that $I \in \mathfrak{X}^{(Y_1)_B}_{\Psi,\Phi,\Theta}$ for I the identity matrix. Hence $(Y_1)_B, (Y_2)_B \in \{\mathrm{Ad}^*(g)(Y_1)_B : g \in \mathfrak{X}^{(Y_1)_B}_{\Psi,\Phi,\Theta}\}$. Since (Ψ, Φ, Θ) is terminal, this set is of size one, so $(Y_1)_B = (Y_2)_B$. Thus for each orbit $[Y]$, there is unique $Y_B \in \mathcal{B}$ such that $[Y] = [Y_B]$. Thus we have a well-defined map π from the set of coadjoint orbits of G to $\mathcal{B}$ where $\pi([Y]) = Y_B$.

Suppose $Y_1, Y_2 \in \mathfrak{l}$ such that $\pi([Y_1]) = Y_B$ and $\pi([Y_2]) = Y_B$. Thus $[Y_1] = [Y_B] = [Y_2]$. Therefore π is one-to-one. Let $Y_B \in \mathcal{B}$. Then there exists terminal $(\Psi, \Phi, \Theta) \in \mathcal{F}$ such that for some $Y \in \mathfrak{l}_{\Psi,\Phi}$, $\mathrm{Ad}^*(g)Y = Y_B$ for all $g \in \mathfrak{X}^{Y}_{\Psi',\Phi',\Theta'}$. Note that $Y \in [Y] \cap \mathfrak{l}_{\Psi,\Phi}$ and thus $\pi([Y]) = Y_B$. Therefore π is onto. Hence π is a one-to-one correspondence between $\mathcal{B}$ and the set of coadjoint orbits of G. $\square$

5. 4×4 EXAMPLE

Consider the case where $n = 4$. Let $\mathfrak{l} = \mathfrak{l}_4$ and $G = G_4$.

Let

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Y_{2,1} & 0 & 0 & 0 \\ Y_{3,1} & Y_{3,2} & 0 & 0 \\ Y_{4,1} & Y_{4,2} & Y_{4,3} & 0 \end{pmatrix} \in \mathfrak{l}$$

and

$$g = \begin{pmatrix} 1 & g_{1,2} & g_{1,3} & g_{1,4} \\ 0 & 1 & g_{2,3} & g_{2,4} \\ 0 & 0 & 1 & g_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G.$$

Then

$$\mathrm{Ad}^*(g)Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ Y_{3,1} - g_{3,4}Y_{4,1} & b & 0 & 0 \\ Y_{4,1} & Y_{4,2} + Y_{4,1}g_{1,2} & c & 0 \end{pmatrix} \tag{9}$$

where

$$a = Y_{2,1} - g_{2,3}Y_{3,1} - g_{2,4}Y_{4,1} + g_{2,3}g_{3,4}Y_{4,1},$$

$$b = Y_{3,2} + Y_{3,1}g_{1,2} - g_{3,4}Y_{4,2} - g_{3,4}Y_{4,1}g_{1,2},$$

$$c = Y_{4,3} + Y_{4,1}g_{1,3} + Y_{4,2}g_{2,3}.$$

Note that for all $g \in G$, $(\mathrm{Ad}^*(g)Y)_{4,1} = Y_{4,1}$. Thus $h \in F[\mathfrak{l}]$ where $h(Y) = Y_{4,1}$ is normal with respect to $(\varnothing, \varnothing, \varnothing)$ because for $f \in F[\mathfrak{l}]$ where $f(Y) = Y_{4,1}$ we have $f(\mathrm{Ad}^*(g)Y) = h(Y) = Y_{4,1}$ for all $(g, Y) \in G \times \mathfrak{l} = \mathfrak{X}_{\varnothing,\varnothing,\varnothing}$. Observe that in this case $f = h$, although this is not true in general. Note that there exist $Y_1, Y_2 \in \mathfrak{l}$ such that $(Y_1)_{4,1} \neq 0$ and $(Y_2)_{4,1} = 0$ which implies that $h(Y_1) \neq 0$ and $h(Y_2) = 0$.

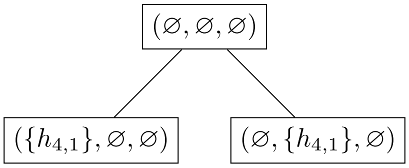

FIGURE 4.

Let $Y_1, Y_2 \in \mathfrak{l}$, such that $h(Y_1) \neq 0$ and $h(Y_2) = 0$ then $(Y_1)_{4,1} \neq 0$ and $(Y_2)_{4,1} = 0$. Thus $(\mathrm{Ad}^*(g)Y_1)_{4,1} \neq 0$ and $(\mathrm{Ad}^*(g)Y_2)_{4,1} = 0$ for all $g \in G$. Hence $[Y_1] \cap [Y_2] = \varnothing$. Therefore h is orbital with respect to $(\varnothing,\varnothing,\varnothing)$. Thus $\{(\{h\},\varnothing,\varnothing),(\varnothing,\{h\},\varnothing)\}$ is a decomposition of $(\varnothing,\varnothing,\varnothing)$. Denote h by $h_{4,1}$. We depict this decomposition in the rooted tree in Figure 4.

Next we consider the triple $(\{h_{4,1}\},\varnothing,\varnothing)$. First we check if $(\{h_{4,1}\},\varnothing,\varnothing)$ is terminal, and if not we seek to find a restriction of $(\{h_{4,1}\},\varnothing,\varnothing)$. Recall that a triple (Ψ,Φ,Θ) is terminal if for all $Y \in \mathfrak{l}_{\Psi,\Phi}$, $\{\mathrm{Ad}^*(g)Y : g \in \mathfrak{X}^Y_{\Psi,\Phi,\Theta}\}$ is of size 1. Note that for $Y \in \mathfrak{l}_{\{h\},\varnothing}$, $\mathfrak{X}^Y_{\{h\},\varnothing,\varnothing} = G$. Thus there exists $g_1, g_2 \in \mathfrak{X}^Y_{\{h\},\varnothing,\varnothing}$ such that $(g_1)_{1,2} \neq 0$ and $(g_2)_{1,2} = 0$. Hence $Y \in \mathfrak{l}_{\{h\},\varnothing}$, $(\mathrm{Ad}^*(g_1)Y)_{4,2} = Y_{4,2} + Y_{4,1}g_{1,2}$ but $(\mathrm{Ad}^*(g_2)Y)_{4,2} = Y_{4,2}$. Since $(\mathrm{Ad}^*(g_1)Y)_{4,2} \neq (\mathrm{Ad}^*(g_2)Y)_{4,2}$ in general, then $\{\mathrm{Ad}^*(g)Y : g \in \mathfrak{X}^Y_{\{h_{4,1}\},\varnothing,\varnothing}\}$ has size of at least 2. Therefore $(\{h_{4,1}\},\varnothing,\varnothing)$ is not terminal.

Since $(\{h_{4,1}\},\varnothing,\varnothing)$ is not terminal, now we look for a restriction. Note that for all $Y \in \mathfrak{l}_{\{h_{4,1}\},\varnothing}$, for $g \in G$ such that

$$g_{1,2} = -\frac{Y_{4,2}}{Y_{4,1}}, \tag{10}$$

by (9) we have $(\mathrm{Ad}^*(g)Y)_{4,2} = 0$. Observe that we know $\frac{Y_{4,2}}{Y_{4,1}}$ is defined since $Y \in \mathfrak{l}_{\{h_{4,1}\}}$, so $h_{4,1}(Y) = Y_{4,1} \neq 0$. Thus $(\{h_{4,1}\},\varnothing,\{(4,2)\})$ is genuine, since

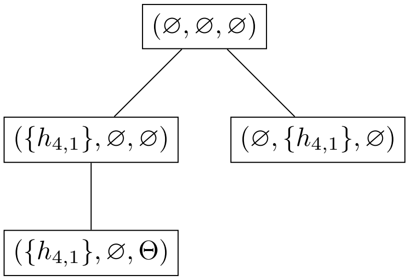

Figure 5.

$\mathfrak{l}_{\{h_{4,1}\},\varnothing} \neq \varnothing$ and for all $Y \in \mathfrak{l}_{\{h_{4,1}\},\varnothing}$ we have $g \in \mathfrak{X}^{Y}_{\{h_{4,1}\},\varnothing,\{(4,2)\}}$ where $g_{1,2} = -\frac{Y_{4,2}}{Y_{4,1}}$. Therefore $(\{h_{4,1}\}, \varnothing, \{(4,2)\})$ is a restriction of $(\{h_{4,1}\}, \varnothing, \varnothing)$. In addition, note that for $Y \in \mathfrak{l}_{\{h_{4,1}\},\varnothing}$ and $g \in G$, by (9),

$$g_{1,3} = \frac{-Y_{4,3} - Y_{4,2}g_{2,3}}{Y_{4,1}} \implies (\mathrm{Ad}^*(g)Y)_{4,3} = 0, \tag{11}$$

$$g_{3,4} = \frac{Y_{3,1}}{Y_{4,1}} \implies (\mathrm{Ad}^*(g)Y)_{3,1} = 0, \tag{12}$$

and

$$g_{2,4} = \frac{Y_{2,1} - g_{2,3}Y_{3,1} + g_{2,3}g_{3,4}Y_{4,1}}{Y_{4,1}} \implies (\mathrm{Ad}^*(g)Y)_{2,1} = 0. \tag{13}$$

Let $\Theta = \{(4,2), (4,3), (3,1), (2,1)\}$. Then $\mathfrak{l}_{\{h_{4,1}\},\varnothing} \neq \varnothing$, and for all $Y \in \mathfrak{l}_{\{h_{4,1}\},\varnothing}$ we have $g \in \mathfrak{X}^{Y}_{\{h_{4,1}\},\varnothing,\Theta}$. Therefore $(\{h_{4,1}\}, \varnothing, \Theta)$ is genuine, and thus a restriction of $(\{h_{4,1}\}, \varnothing, \varnothing)$. Hence we will add $(\{h_{4,1}\}, \varnothing, \Theta)$ to our tree at depth 2, giving us the tree in Figure 5.

Continuing down this branch of the tree, we determine if $(\{h_{4,1}\}, \varnothing, \Theta)$ is terminal, and otherwise if there exists a decomposition of $(\{h_{4,1}\}, \varnothing, \Theta)$. If there exists a further restriction, we would group this restriction together with the previous restriction so that at each step we specify as many zero entries in $\mathrm{Ad}^*(g)Y$ as possible.

When we substitute the expressions for $g_{1,2}, g_{1,3}, g_{3,4}$ and $g_{2,4}$ in (10),(11),(12),(13) we find that for $(g, Y) \in \mathfrak{X}_{\{h_{4,1}\},\varnothing,\Theta}$ we have

$$\mathrm{Ad}^*(g)Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{Y_{4,1}Y_{3,2}-Y_{3,1}Y_{4,2}}{Y_{4,1}} & 0 & 0 \\ Y_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Thus for all $Y \in \mathfrak{l}_{\{h_{4,1}\},\varnothing}$, the set $\{\mathrm{Ad}^*(g)Y : g \in \mathfrak{X}^Y_{\{h_{4,1}\},\varnothing,\Theta}\}$ is of size 1, with $\mathrm{Ad}^*(g)Y$ given by the equation above. Hence $(\{h_{4,1}\}, \varnothing, \Theta)$ is terminal.

Now we consider the $(\varnothing, \{h_{4,1}\}, \varnothing)$ node of our tree. Note that for $(g, Y) \in \mathfrak{X}_{\varnothing,\{h_{4,1}\},\varnothing}$ we have $h_{4,1}(Y) = Y_{4,1} = 0$. When we substitute $Y_{4,1} = 0$ into (9) we find that

$$\mathrm{Ad}^*(g)Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Y_{2,1} - g_{2,3}Y_{3,1} & 0 & 0 & 0 \\ Y_{3,1} & Y_{3,2} + Y_{3,1}g_{1,2} - g_{3,4}Y_{4,2} & 0 & 0 \\ 0 & Y_{4,2} & Y_{4,3} + Y_{4,2}g_{2,3} & 0 \end{pmatrix}. \tag{14}$$

Note that for $Y \in \mathfrak{l}_{\varnothing,\{h_{4,1}\}}$, $\mathfrak{X}^Y_{\varnothing,\{h_{4,1}\},\varnothing} = G$. Thus there exists $g_1, g_2 \in \mathfrak{X}^Y_{\varnothing,\{h_{4,1}\},\varnothing}$ such that $(g_1)_{2,3} \neq 0$ and $(g_2)_{2,3} = 0$. Hence for $Y \in \mathfrak{l}_{\varnothing,\{h_{4,1}\}}$, $(\mathrm{Ad}^*(g_1)Y)_{4,3} = Y_{4,3} + Y_{4,2}g_{2,3}$ but $(\mathrm{Ad}^*(g_2)Y)_{4,3} = Y_{4,3}$. Since $(\mathrm{Ad}^*(g_1)Y)_{4,2} \neq (\mathrm{Ad}^*(g_2)_{4,2})$ in general, then $\{\mathrm{Ad}^*(g)Y : g \in \mathfrak{X}^Y_{\varnothing,\{h_{4,1}\},\varnothing}\}$ has size of at least 2. Therefore $(\varnothing, \{h_{4,1}\}, \varnothing)$ is not terminal.

We look for a restriction of $(\varnothing, \{h_{4,1}\}, \varnothing)$. Note that for $Y \in \mathfrak{l}_{\varnothing,\{h_{4,1}\}}$, in order to choose $g \in G$ such that $(\mathrm{Ad}^*(g)Y)_{t,s} = 0$ for some $(t, s) \in M$, we must require that either $Y_{4,2} \neq 0$ or $Y_{3,1} \neq 0$. Thus without further constraining our space $\mathfrak{l}_{\varnothing,\{h_{4,1}\}}$,

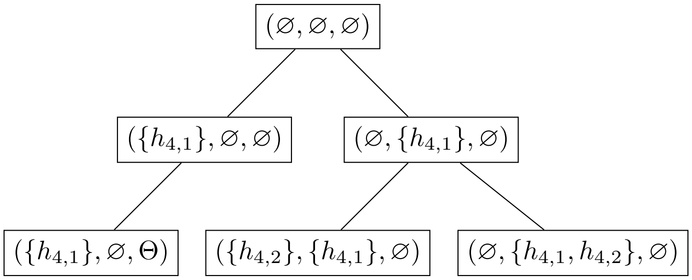

FIGURE 6.

we cannot find a restriction of $(\varnothing, \{h_{4,1}\}, \varnothing)$. Hence we find a decomposition of $(\varnothing, \{h_{4,1}\}, \varnothing)$, which will further constrain $\mathfrak{l}_{\varnothing, \{h_{4,1}\}}$.

Note that for all $(g, Y) \in \mathfrak{X}_{\varnothing, \{h_{4,1}\}, \varnothing}$, $(\mathrm{Ad}^*(g)Y)_{4,2} = Y_{4,2}$. Thus $h_{4,2} \in F[\mathfrak{l}]$ where $h_{4,2}(Y) = Y_{4,2}$ is normal with respect to $(\varnothing, \{h_{4,1}\}, \varnothing)$ because for $f \in F[\mathfrak{l}]$ where $f(Y) = Y_{4,2}$ we have $f(\mathrm{Ad}^*(g)Y) = Y_{4,2} = h_{4,2}(Y)$ for all $(g, Y) \in \mathfrak{X}_{\varnothing, \{h_{4,1}\}, \varnothing}$. Note that for $Y_1, Y_2 \in \mathfrak{l}_{\varnothing, \{h_{4,1}\}}$ if $h_{4,2}(Y_1) \neq 0$ and $h_{4,2}(Y_2) = 0$ then $(Y_1)_{4,2} \neq 0$ and $(Y_2)_{4,2} = 0$. Hence by (14), $(\mathrm{Ad}^*(g_1)Y_1)_{4,2} \neq 0$ and $(\mathrm{Ad}^*(g_2)Y_2)_{4,2} = 0$ for all $g_1, g_2 \in G$. Thus $[Y_1] \cap [Y_2] = \varnothing$. Note also that there exists $Y_1, Y_2 \in \mathfrak{l}_{\varnothing, \{h_{4,1}\}}$ such that $(Y_1)_{4,2} \neq 0$ and $(Y_2)_{4,2} = 0$ which implies that $h_{4,2}(Y_1) \neq 0$ and $h_{4,2}(Y_2) = 0$. Therefore $h_{4,2}$ is orbital with respect to $(\varnothing, \{h_{4,1}\}, \varnothing)$. Hence $\{(\{h_{4,2}\}, \{h_{4,1}\}, \varnothing), (\varnothing, \{h_{4,1}, h_{4,2}\}, \varnothing)\}$ is a decomposition of $(\varnothing, \{h_{4,1}\}, \varnothing)$. When we add these triples to the tree we get Figure 6.

Now consider the triple $(\{h_{4,2}\}, \{h_{4,1}\}, \varnothing)$. For $(g, Y) \in \mathfrak{X}_{\{h_{4,2}\}, \{h_{4,1}\}, \varnothing}$ we have $\mathrm{Ad}^*(g)Y$ given by (14). Note that for some $Y \in \mathfrak{l}_{\{h_{4,2}\}, \{h_{4,1}\}}$ there exists $g_1, g_2 \in \mathfrak{X}^Y_{\{h_{4,2}\}, \{h_{4,1}\}, \varnothing}$ such that $(g_1)_{2,3} \neq (g_2)_{2,3}$ and as a result $(\mathrm{Ad}^*(g_1)Y)_{4,3} \neq$

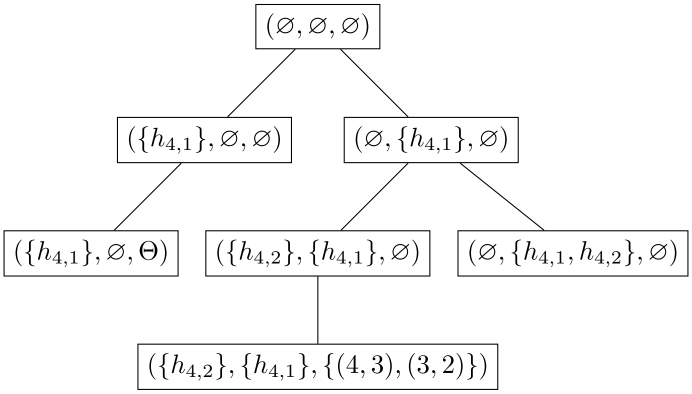

Figure 7.

$(\mathrm{Ad}^*(g_2)Y)_{4,3}$. Hence $(\{h_{4,2}\}, \{h_{4,1}\}, \varnothing)$ is not terminal. Next we look for a restriction of $(\{h_{4,2}\}, \{h_{4,1}\}, \varnothing)$. Note that for $(g, Y) \in \mathfrak{X}_{\{h_{4,2}\},\{h_{4,1}\},\varnothing}$,

$$g_{2,3} = -\frac{Y_{4,3}}{Y_{4,2}} \implies (\mathrm{Ad}^*(g)Y)_{4,3} = 0$$

and

$$g_{3,4} = \frac{Y_{3,2} + Y_{3,1}g_{1,2}}{Y_{4,2}} \implies (\mathrm{Ad}^*(g)Y)_{3,2} = 0.$$

Since $Y \in \mathfrak{l}_{\{h_{4,2}\},\{h_{4,1}\}}$ then $Y_{4,2} \neq 0$, so these expressions are defined. Thus for each $Y \in \mathfrak{l}_{\{h_{4,2}\},\{h_{4,1}\}}$ there exists $g \in \mathfrak{X}^Y_{\{h_{4,2}\},\{h_{4,1}\},\{(4,3),(3,2)\}}$. Since we also know $\mathfrak{l}_{\{h_{4,2}\},\{h_{4,1}\}} \neq \varnothing$, then this means $(\{h_{4,2}\}, \{h_{4,1}\}, \{(4,3), (3,2)\})$ is genuine. Therefore $(\{h_{4,2}\}, \{h_{4,1}\}, \{(4,3), (3,2)\})$ is a restriction of $(\{h_{4,2}\}, \{h_{4,1}\}, \varnothing)$. Thus we add $(\{h_{4,2}\}, \{h_{4,1}\}, \{(4,3), (3,2)\})$ to our tree to get Figure 7.

Now we consider the node $(\{h_{4,2}\}, \{h_{4,1}\}, \{(4,3), (3,2)\})$. By substituting the expressions found for $g_{2,3}$ and $g_{3,4}$ above into (14) we can find an equation for

$\mathrm{Ad}^*(g)Y$ for all $(g,Y) \in \mathfrak{X}_{\{h_{4,2}\},\{h_{4,1}\},\{(4,3),(3,2)\}}$. When we do this we find

$$\mathrm{Ad}^*(g)Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Y_{2,1} + \frac{Y_{4,3}Y_{3,1}}{Y_{4,2}} & 0 & 0 & 0 \\ Y_{3,1} & 0 & 0 & 0 \\ 0 & Y_{4,2} & 0 & 0 \end{pmatrix}. \tag{15}$$

Thus for all $Y \in \mathfrak{l}_{\{h_{4,2}\},\{h_{4,1}\}}$, we have $\{\mathrm{Ad}^*(g)Y : g \in \mathfrak{X}^{Y}_{\{h_{4,2}\},\{h_{4,1}\},\{(4,3),(3,2)\}}\}$ is of size 1 with $\mathrm{Ad}^*(g)Y$ given by (15). Therefore $(\{h_{4,2}\},\{h_{4,1}\},\{(4,3),(3,2)\})$ is terminal.

Next we consider the node $(\varnothing,\{h_{4,1},h_{4,2}\},\varnothing)$. Note that for $Y \in \mathfrak{l}_{\varnothing,\{h_{4,1},h_{4,2}\}}$, $Y_{4,1} = Y_{4,2} = 0$. Upon substituting $Y_{4,2} = 0$ into (14) we find that for $(g,Y) \in \mathfrak{X}_{\{h_{4,2}\},\{h_{4,1}\},\varnothing}$,

$$\mathrm{Ad}^*(g)Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Y_{2,1} - g_{2,3}Y_{3,1} & 0 & 0 & 0 \\ Y_{3,1} & Y_{3,2} + Y_{3,1}g_{1,2} & 0 & 0 \\ 0 & 0 & Y_{4,3} & 0 \end{pmatrix}. \tag{16}$$

Note that for some $Y \in \mathfrak{l}_{\varnothing,\{h_{4,1},h_{4,2}\}}$ there exists $g_1, g_2 \in \mathfrak{X}^{Y}_{\varnothing,\{h_{4,1},h_{4,2}\},\varnothing}$ such that $(g_1)_{1,2} \neq (g_2)_{1,2}$ and as a result $(\mathrm{Ad}^*(g_1)Y)_{3,2} \neq (\mathrm{Ad}^*(g_2)Y)_{3,2}$. Hence $(\varnothing,\{h_{4,1},h_{4,2}\},\varnothing)$ is not terminal.

Since $(\varnothing,\{h_{4,1},h_{4,2}\},\varnothing)$ is a decomposition of $(\varnothing,\{h_{4,1}\},\varnothing)$, next we look for a restriction of $(\varnothing,\{h_{4,1},h_{4,2}\},\varnothing)$. We can see from (16) that to make any more entries of $(\mathrm{Ad}^*(g)Y)$ equal to 0 we must have $Y_{3,1} \neq 0$. Since we do not know this in general for $Y \in \mathfrak{l}_{\varnothing,\{h_{4,1},h_{4,2}\}}$, we cannot find a restriction of $(\varnothing,\{h_{4,1},h_{4,2}\},\varnothing)$.

Thus we look for another decomposition. Note that for all $(g,Y) \in \mathfrak{X}_{\varnothing,\{h_{4,1},h_{4,2}\},\varnothing}$, $(\mathrm{Ad}^*(g)Y)_{3,1} = Y_{3,1}$. Thus $h_{3,1} \in F[\mathfrak{l}]$ such that $h_{3,1}(Y) = Y_{3,1}$ is normal with

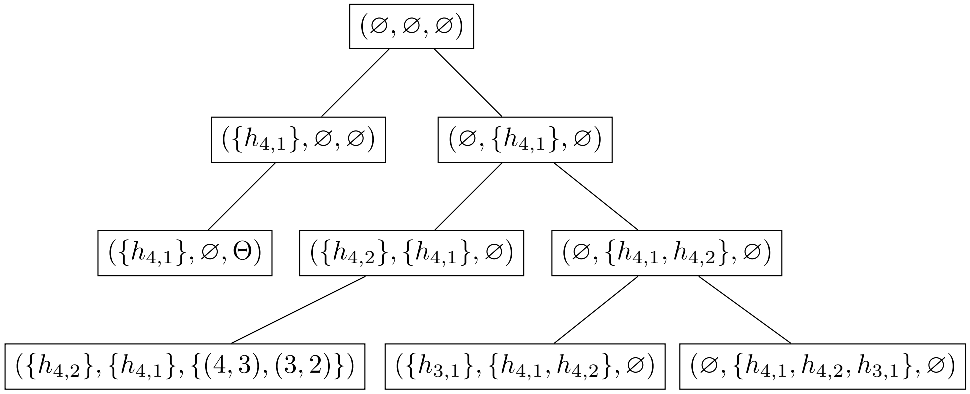

FIGURE 8.

respect to $(\varnothing, \{h_{4,1}, h_{4,2}\}, \varnothing)$ since for $f \in F[\mathfrak{l}]$ such that $f(Y) = Y_{3,1}$ we have $f(\mathrm{Ad}^*(g)Y) = Y_{3,1} = h_{3,1}(Y)$. Note that there exists $Y_1, Y_2 \in \mathfrak{l}_{\varnothing,\{h_{4,1},h_{4,2}\}}$ such that $h_{3,1}(Y_1) \neq 0$ and $h_{3,1}(Y_2) = 0$. For such Y_1, Y_2 we have that $(\mathrm{Ad}^*(g)Y_1)_{3,1} \neq 0$ and $(\mathrm{Ad}^*(g)Y_2)_{3,1} = 0$ for all $g \in G$. Thus $[Y_1] \cap [Y_2] = \varnothing$. Therefore $h_{3,1}$ is orbital with respect to $(\varnothing, \{h_{4,1}, h_{4,2}\}, \varnothing)$, so $\{(\{h_{3,1}\}, \{h_{4,1}, h_{4,2}\}, \varnothing), (\varnothing, \{h_{4,1}, h_{4,2}, h_{3,1}\}, \varnothing)\}$ is a decomposition of $(\varnothing, \{h_{4,1}, h_{4,2}\}, \varnothing)$. When we add this to our tree we get Figure 8.

Next we will consider the triple $(\{h_{3,1}\}, \{h_{4,1}, h_{4,2}\}, \varnothing)$. Note that $\mathrm{Ad}^*(g)Y$ for $(g, Y) \in \mathfrak{X}$ is given by (16). Observe that for $Y \in \mathfrak{l}_{\Psi,\Phi}$, if we have $g \in \mathfrak{X}^Y_{\Psi,\Phi,\Theta}$ such that $g_{2,3} = \frac{Y_{2,1}}{Y_{3,1}}$ and $g_{1,2} = -\frac{Y_{3,2}}{Y_{3,1}}$ then $(\mathrm{Ad}^*(g)Y)_{2,1} = (\mathrm{Ad}^*(g)Y)_{3,2} = 0$. Thus for $Y \in \mathfrak{l}_{\{h_{3,1}\},\{h_{4,1},h_{4,2}\}}$, $\mathfrak{X}^Y_{\{h_{3,1}\},\{h_{4,1},h_{4,2}\},\{(2,1),(3,2)\}} \neq \varnothing$. Since $\mathfrak{l}_{\{h_{3,1}\},\{h_{4,1},h_{4,2}\}} \neq \varnothing$, then $(\{h_{3,1}\}, \{h_{4,1}, h_{4,2}\}, \{(2,1),(3,2)\})$ is genuine. Thus $(\{h_{3,1}\}, \{h_{4,1}, h_{4,2}\}, \{(2,1),(3,2)\})$ is a restriction of $(\{h_{3,1}\}, \{h_{4,1}, h_{4,2}\}, \varnothing)$. We depict this restriction in the rooted tree in Figure 9.

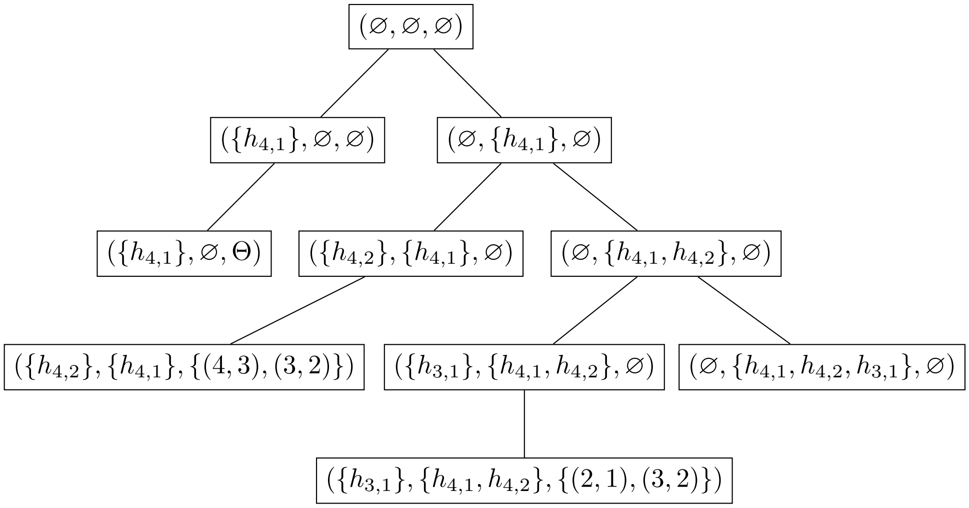

FIGURE 9.

Consider the node $(\{h_{3,1}\}, \{h_{4,1}, h_{4,2}\}, \{(2,1),(3,2)\})$. Note that for $(g, Y) \in \mathfrak{X}_{\{h_{3,1}\},\{h_{4,1},h_{4,2}\},\{(2,1),(3,2)\}}$ we can find an equation for $\mathrm{Ad}^*(g)Y$ for by substituting the expressions for $g_{2,3}$ and $g_{1,2}$ found above into (16). This gives us that

$$\mathrm{Ad}^*(g)Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y_{3,1} & 0 & 0 & 0 \\ 0 & 0 & Y_{4,3} & 0 \end{pmatrix}.$$

This demonstrates that for all $Y \in \mathfrak{l}_{\{h_{3,1}\},\{h_{4,1},h_{4,2}\}}$, $\{\mathrm{Ad}^*(g)Y : g \in \mathfrak{X}^{Y}_{\{h_{3,1}\},\{h_{4,1},h_{4,2}\},\{(2,1),(3,2)\}}\}$ is of size 1. Therefore $(\{h_{3,1}\}, \{h_{4,1}, h_{4,2}\}, \{(2,1),(3,2)\})$ is terminal.

Finally, consider $(\varnothing, \{h_{4,1}, h_{4,2}, h_{3,1}\}, \varnothing)$. When we substitute $Y_{4,1} = Y_{4,2} = Y_{3,1} = 0$ we find that for $(g, Y) \in \mathfrak{X}_{\varnothing,\{h_{4,1},h_{4,2},h_{3,1}\},\varnothing}$,

$$\mathrm{Ad}^*(g)Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Y_{2,1} & 0 & 0 & 0 \\ 0 & Y_{3,2} & 0 & 0 \\ 0 & 0 & Y_{4,3} & 0 \end{pmatrix}.$$

Thus we can see that $(\varnothing, \{h_{4,1}, h_{4,2}, h_{3,1}\}, \varnothing)$ is terminal. Therefore all branches in our rooted tree end in a terminal node, so our classification is complete.

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