VARIATIONS ON MULTIPLICATIVE PRESERVERS OF THE C-NUMERICAL RANGE AND RADIUS

by

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Submitted in partial fulfillment of the requirements for Major Honors in Mathematics

Houghton College, Houghton, NY

April 2015

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1 Introduction

This paper will look at functions that preserve the C-numerical range or radius, which are specific sets associated with a matrix. The study of numerical ranges is important as numerical ranges have many applications in mathematics, including operator theory and perturbation theory. There are also some other areas of science that study generalizations of the C-numerical range. Since all of the eigenvalues of a matrix are contained within numerical ranges, engineers use the numerical ranges to help approximate eigenvalues. Also, in quantum computing numerical ranges are used by generalizing the numerical ranges to Krein spaces.

The *C-numerical range* is a subset of the complex plane associated with a matrix. The *C-numerical radius* in turn takes the C-numerical range and associates it with a non-negative real number. The C-numerical range and radius are well-studied structures that capture information about the matrix as a linear transformation. One way to garner information about the C-numerical range and radius is by studying functions that preserve some aspect of the C-numerical range or radius. Researchers who work with numerical ranges or radii frequently study specific types of preserving functions called multiplicative preservers. A multiplicative preserver is a preserving function ϕ on the set of all square matrices of a fixed size such that $\phi(AB) = \phi(A)\phi(B)$ for all matrices A and B. A function is called a multiplicative preserver of the C-numerical range or radius if it is multiplicative and both A and $\phi(A)$ have the same C-numerical range or radius for any invertible matrix A.

Our goal is to classify all possible forms of these multiplicative maps that preserve the C-numerical range or radius. The classification of these multiplicative preservers of the C-numerical range and radius in turn gives a better understanding of the interaction between a matrix, its C-numerical range or radius, and multiplicative maps of that matrix. Inspired by a talk by Chi-Kwong Li (see [5]), we explore different binary operations on two matrices and the implications in relation to the forms of the C-numerical range and radius preserving maps.

2 Background

The study of multiplicative preservers of the C-numerical range and radius requires knowledge of complex numbers, linear algebra, and group theory. We will assume a linear algebra background and cover the basic definitions needed from group theory and complex numbers to work with the C-numerical range and radius. See [1] for a linear algebra reference text and [4] for an algebra reference text. We will then survey existing results related to multiplicative preservers of the C-numerical range and radius.

2.1 Basic Definitions

Let X be a set. A *binary operation* is a function $* : X \times X \to X$ where $X \times X$ is the set of all ordered pairs (a, b) with $a, b \in X$. A *group* is a set G with a binary operation $*$ such that the following properties hold given any $a, b, c \in G$:

Closure:	$a * b \in G$,
Associativity:	$(a * b) * c = a * (b * c)$,
Identity:	there exists an element $e \in G$ such that $a * e = e * a = a$, and
Inverse:	there exists an element $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$.

As a motivating example of a group, we will consider the set of all invertible 2×2 matrices with complex entries, denoted $GL_2(\mathbb{C})$, under the operation of ordinary matrix multiplication. We note that $GL_2(\mathbb{C})$ is closed under matrix multiplication since the product of two 2×2 matrices with complex entries is a 2×2 matrix with complex entries. We also note matrix multiplication is associative. For the identity element, we have $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $\det(A)$ denotes the determinant. Hence we have that $GL_2(\mathbb{C})$ forms a group under matrix multiplication, which we will denote $(GL_2(\mathbb{C}), \cdot)$.

Given groups $(G, *)$ and $(H, \cdot)$, a function $\phi : (G, *) \to (H, \cdot)$ is a *group homomorphism* if and only if for all $g, h \in G$,

$$\phi(g * h) = \phi(g) \cdot \phi(h).$$

We use complex numbers throughout, so let us recall some basic properties of the set of complex numbers. We can represent a complex number $z \in \mathbb{C}$ as the sum $z = x + iy$ for x and y real numbers. The *complex conjugate* of a complex number z is $\overline{z} = x - iy$. Geometrically, we can represent complex numbers as points in $\mathbb{R}^2$ with the first coordinate being the real part and the second coordinate being the imaginary part of the complex number. The distance from the origin to that point is the *modulus* of the complex number, so for $z = x+iy$, the modulus is $|z| = \sqrt{x^2 + y^2}$. From these definitions, we deduce $z\overline{z} = x^2 + y^2 = |z|^2$. Thus the inverse of a nonzero z is $z^{-1} = \frac{\overline{z}}{|z|^2}$.

Now given a matrix A, we denote the transpose by A^T. We also use $\overline{A}$ to represent the complex conjugate of A, which takes the complex conjugate of each entry in A. The conjugate transpose of A is $A^* = \left(\overline{A}\right)^T$. Also the trace of a matrix A is denoted

$\text{tr}(A)$.

We let the set of all $n \times n$ matrices with complex entries be $M_n(\mathbb{C})$. The subset of $M_n(\mathbb{C})$ consisting of all invertible matrices, that is, matrices with nonzero determinant, is called the *general linear group* and denoted $GL_n(\mathbb{C})$. The subset of $GL_n(\mathbb{C})$ consisting of all matrices of determinant 1 or -1 is the *special linear group* denoted $SL_n(\mathbb{C})$. Finally, a matrix $U \in GL_n(\mathbb{C})$ is *unitary* if and only if $U^{-1} = U^*$. The set of all unitary matrices with complex entries is denoted $\mathcal{U}_n(\mathbb{C})$.

We now turn our attention to the C-numerical range and radius. The *C-numerical range* of a matrix $A \in M_n(\mathbb{C})$ for some matrix $C \in M_n(\mathbb{C})$ is given by

$$W_C(A) = \{\text{tr}(CUAU^*) : U \in \mathcal{U}_n(\mathbb{C})\}.$$

The *C-numerical radius* of a matrix $A \in M_n(\mathbb{C})$ is given by

$$w_C(A) = \max\{|z| : z \in W_C(A)\}.$$

As an example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in GL_2(\mathbb{C})$, and let $C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. In order to find $W_C(A)$ explicitly, we note that any 2×2 unitary matrix is of the form $U = \begin{bmatrix} u_1 & u_2 \\ \overline{u_2} & -\overline{u_1} \end{bmatrix}$, where $|u_1|^2 + |u_2|^2 = 1$. Thus we find

$$\begin{aligned}
W_C(A) &= \{\text{tr}(CUAU^*) : U \in \mathcal{U}_2(\mathbb{C})\} \\
&= \left\{ \text{tr}\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ \overline{u_2} & -\overline{u_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \overline{u_1} & u_2 \\ \overline{u_2} & -u_1 \end{bmatrix} \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ 3 \cdot \text{tr}\left(\begin{bmatrix} u_1 & 2u_2 \\ \overline{u_2} & -2\overline{u_1} \end{bmatrix} \begin{bmatrix} \overline{u_1} & u_2 \\ \overline{u_2} & -u_1 \end{bmatrix} \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ 3 \cdot \text{tr}\left(\begin{bmatrix} |u_1|^2 + 2|u_2|^2 & -u_1 u_2 \\ -\overline{u_1 u_2} & 2|u_1|^2 + |u_2|^2 \end{bmatrix} \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ 3 \cdot \left(|u_1|^2 + 2|u_2|^2 + 2|u_1|^2 + |u_2|^2 \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ 9 \cdot \left((|u_1|^2 + |u_2|^2) \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \{9\}.
\end{aligned}$$

Thus we have

$$w_C(A) = \max\{|z| : z \in W_C(A)\} = \max\{|z| : z \in \{9\}\} = 9.$$

The previous example gives us an idea of what it is like to work with the C-numerical range and radius. However, we cannot always explicitly compute the C-numerical range or radius because we do not have a simple form for unitary matrices larger than 2×2 and the computations become more difficult as C and A become more complicated.

A map $\phi : (GL_n(\mathbb{C}), \cdot) \to (M_n(\mathbb{C}), \cdot)$ is *multiplicative preserver* of the C-numerical range [resp. radius] if and only if ϕ is a group homomorphism and ϕ preserves the C-numerical range [radius]. That is, given any $A, B \in GL_n(\mathbb{C})$, $\phi(AB) = \phi(A)\phi(B)$, and $W_C(\phi(A)) = W_C(A)$ $[w_C(\phi(A)) = w_C(A)]$. Our overarching goal is to classify all multiplicative maps that preserve the C-numerical range or radius.

For our final example to understand our definitions, consider $\phi : GL_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by $\phi(A) = VAV^*$ for some unitary matrix $V \in GL_2(\mathbb{C})$. Now we claim that multiplying V by the set of unitary matrices, $\mathcal{U}_2(\mathbb{C})$, we still have the entire set of unitary matrices. To see this, consider $V\mathcal{U}_2(\mathbb{C}) = \{VU : U \in \mathcal{U}_2(\mathbb{C})\}$. We first show that $\mathcal{U}_2(\mathbb{C})$ and $V\mathcal{U}_2(\mathbb{C})$ have the same cardinality. To see this, suppose $VW = VU$ for some $W, U \in \mathcal{U}_2(\mathbb{C})$. Then since V is invertible, $W = U$, and no two elements in $\mathcal{U}_2(\mathbb{C})$ are mapped to the same matrix when multiplied by V. Next we claim that each of the elements of $V\mathcal{U}_2(\mathbb{C})$ is unitary. Let $VU \in V\mathcal{U}_2(\mathbb{C})$. Then

$$(VU)^{-1} = U^{-1}V^{-1} = U^*V^* = \left(\overline{U}\right)^T \left(\overline{V}\right)^T = \left(\overline{VU}\right)^T = (VU)^*,$$

and VU is unitary as desired. Thus we have a set of unitary matrices with the same number of elements as the set $\mathcal{U}_2(\mathbb{C})$, and we conclude $V\mathcal{U}_2(\mathbb{C}) = \mathcal{U}_2(\mathbb{C})$.

Using the fact that multiplying a unitary matrix by the set of unitary matrices still gives all unitary matrices, we have

$$W_C(\phi(A)) = \{\text{tr}(CUVAV^*U^* : U \in \mathcal{U}_2(\mathbb{C})\} = \{\text{tr}(CWAW^* : W \in \mathcal{U}_2(\mathbb{C})\} = W_C(A).$$

Since ϕ preserves the C-numerical range, ϕ must also preserve the C-numerical radius of A. We also note ϕ is multiplicative as, given any $B, C \in GL_2(\mathbb{C})$,

$$\phi(BC) = VBCV^* = VB(V^*V)CV^* = (VBV^*)(VCV^*) = \phi(B)\phi(C).$$

Therefore ϕ is a multiplicative preserver of both the C-numerical range and the C-numerical radius.

Now that we have an understanding of the C-numerical range and radius as well as multiplicative preservers of these functions, in the next section we will survey the theorems that allow us to fully classify the set of multiplicative preservers of the C-numerical range and radius.

2.2 Theorems

We will now give a survey of the main theorems related to multiplicative preservers of the C-numerical range and radius. We will use the following notation:

$$\mathcal{H} = SL_n(\mathbb{C}) \text{ or } GL_n(\mathbb{C})$$
$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$$
$$S^1 = \text{ the unit circle in } \mathbb{C} = \{z \in \mathbb{C} : |z| = 1\}$$
$$\overline{A} = (\overline{a_{ij}})_n \text{ for } A = (a_{ij})_n \in M_n(\mathbb{C})$$
$$A^T = (a_{ij})_n^T = (a_{ji})_n \text{ for } A = (a_{ij})_n \in M_n(\mathbb{C})$$
$$A^* = \overline{A}^T \text{ for } A \in M_n(\mathbb{C})$$

We begin with a full classification of all nontrivial multiplicative maps on $SL_n(\mathbb{C})$. The problem is addressed fully for any group in [3], but we will only state the results in terms of our groups $\mathcal{H}$ under matrix multiplication. All of the results take the form of conjugates. Conjugation is the process of multiplying a matrix by another matrix on the left and the inverse of the other matrix on the right. Thus given $A \in M_n(\mathbb{C})$ and $X \in GL_n(\mathbb{C})$, conjugation of A by X is given by XAX^{-1}. We say matrices A and B are similar matrices if there exists a matrix $X \in GL_n(\mathbb{C})$ such that $B = XAX^{-1}$. Thus, the process of conjugating A is called a similarity transform. Such transforms may or may not change the numerical range (see section 5 in [9]). All of the following forms are some variation of a similarity transform. From Theorem 2.5 of [3] we obtain the following theorem concerning $SL_n(\mathbb{C})$:

Theorem 1. *Let $\phi : SL_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a nontrivial multiplicative map. Then, ϕ is a group homomorphism into $SL_n(\mathbb{C})$ such that ϕ has one of the following forms for some $S \in GL_n(\mathbb{C})$:*

1. $A \mapsto SAS^{-1}$,
2. $A \mapsto S\overline{A}S^{-1}$,
3. $A \mapsto SA^*S^{-1}$, *or*
4. $A \mapsto SA^TS^{-1}$.

For $GL_n(\mathbb{C})$, Guralanik, et. al. classify all multiplicative maps in Theorem 2.7 of [3]. In the more general setting of $GL_n(\mathbb{C})$, these multiplicative maps can also have special scalar multiples of the similarity transforms in the previous theorem.

Theorem 2. *Let $\phi : GL_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a multiplicative map. Then, there exists a mapping $f : \mathbb{C}^* \to \mathbb{C}^*$ where $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$, and $S \in GL_n(\mathbb{C})$ such that ϕ has one of the following forms:*

1. $A \mapsto f(\det(A))SAS^{-1}$,

2. $A \mapsto f(\det(A))S\overline{A}S^{-1}$,

3. $A \mapsto f(\det(A))SA^*S^{-1}$, *or*

4. $A \mapsto f(\det(A))SA^TS^{-1}$.

Thus if we have a multiplicative map that preserves the C-numerical range or radius, we know it must have one of the above forms because it is multiplicative. Since each of these forms is multiplicative, our goal is to decide if these forms necessarily preserve the C-numerical range and radius for all $A \in GL_n(\mathbb{C})$. In other words, if we have a map that is multiplicative, it must have one of the above forms. If we add the restriction that our multiplicative map also preserve the C-numerical range or radius, we can then ask which of the forms remain.

We first consider the case where C is a *scalar matrix*, which is a matrix that can be represented as μI where $\mu \in \mathbb{C}$. If $\mu = 0$, then $C = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ and for any $M \in \mathcal{H}$

$$W_C(M) = \{\mathrm{tr}(CUMU^*) : U \in \mathcal{U}_n(\mathbb{C})\} = \left\{ \mathrm{tr} \left(\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \right) \right\} = \{0\}.$$

Thus, $W_C(A) = 0 = W_C(\phi(A))$ since both A and $\phi(A)$ are in $M_n(\mathbb{C})$. Therefore the multiplicative preservers of the C-numerical range are exactly the multiplicative maps in Theorems 1 and 2 when C is the zero matrix.

For the next part, we note that the trace is invariant under certain cyclic permutations. These permutations follow from Proposition 10.9 of [1] which states $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for matrices $A, B \in M_n(\mathbb{C})$. A permutation of multiplied matrices is a rearrangement of the order in which the matrices are multiplied. For example a permutation of the product AB can be given by the product BA. A trace invariant cyclic permutation is a permutation that can be thought of as putting beads on a necklace with a different color corresponding to each matrix in the product. Then by sliding the beads around on the necklace, any way we slide the beads will still result in a cyclic permutation. As an example, consider the product ABC with beads labeled A, B, and C, respectively. We then slide bead A around so our beads are now in the order bead B, bead C, and bead A. Thus BCA is a cyclic permutation of ABC and $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$. Using this method we can conclude $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$. However, without breaking the necklace, we cannot slide the beads so that they are in the order bead B, bead A, and bead C. Thus the permutation BAC is not a cyclic permutation and it is not necessarily the case that $\text{tr}(ABC) = \text{tr}(BAC)$.

Next consider $C = \mu I$ for some $\mu \in \mathbb{C}^*$. For any $A \in \mathcal{H}$,

$$\begin{aligned} W_C(\phi(A)) &= \{\text{tr}(CU\phi(A)U^*) : U \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{\text{tr}(\mu U\phi(A)U^*) : U \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{\mu\text{tr}(U\phi(A)U^*) : U \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{\mu\text{tr}(\phi(A)U^*U) : U \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{\mu\text{tr}(\phi(A))\}, \end{aligned} \tag{1}$$

and we also have

$$\begin{aligned} W_C(A) &= \{\text{tr}(CUAU^*) : U \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{\mu\text{tr}(AU^*U) : U \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{\mu\text{tr}(A)\}. \end{aligned} \tag{2}$$

Hence, the multiplicative preservers that also preserve the C-numerical range are reduced to multiplicative maps that preserve the trace. In other words, since $\mu \neq 0$, we can divide $\mu\text{tr}(\phi(A)) = \mu\text{tr}(A)$ by μ to find $W_C(\phi(A)) = W_C(A)$ if and only if $\text{tr}(\phi(A)) = \text{tr}(A)$. This case is addressed in Theorem 3.7 of [3], which states

Theorem 3. *A multiplicative map $\phi : \mathcal{H} \to M_n(\mathbb{C})$ satisfies $\text{tr}(\phi(A)) = \text{tr}(A)$ for all $A \in \mathcal{H}$ if and only if there is an $S \in SL_n(\mathbb{C})$ such that*

1. *ϕ has the form $A \mapsto SAS^{-1}$, or*
2. *if $n = 2$, and $\mathcal{H} = SL_2(\mathbb{C})$, ϕ has the form $A \mapsto S(A^{-1})^T S^{-1}$.*

Still considering the case where $C = \mu I$ for $\mu \in \mathbb{C}^*$, the C-numerical radius is given by

$$w_C(A) = \max\{|z| : z \in W_C(A)\} = |\mu\text{tr}(A)| = |\mu| \cdot |\text{tr}(A)|,$$

and

$$w_C(\phi(A)) = \max\{|z| : z \in W_C(\phi(A))\} = |\mu\text{tr}(\phi(A))| = |\mu| \cdot |\text{tr}(\phi(A))|.$$

Thus the multiplicative preservers of the C-numerical radius are given by the multiplicative preservers of the modulus of the trace which is again addressed in [3], giving the following forms:

Theorem 4. *A multiplicative map $\phi : \mathcal{H} \to M_n(\mathbb{C})$ satisfies $|\text{tr}(\phi(A))| = |\text{tr}(A)|$ for all $A \in \mathcal{H}$ if and only if there is an $S \in SL_n(\mathbb{C})$ such that*

1. *ϕ has the form $A \mapsto SAS^{-1}$ or $A \mapsto S\overline{A}S^{-1}$,*

2. *if $n = 2$, and $\mathcal{H} = SL_2(\mathbb{C})$, ϕ has the form $A \mapsto S(A^{-1})^T S^{-1}$, or*

3. *if $\mathcal{H} = GL_n(\mathbb{C})$, for a map $f : \mathbb{C}^* \to S^1$ where $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$, ϕ has the form $A \mapsto f(\det(A))SAS^{-1}$ or $A \mapsto f(\det(A))S\overline{A}S^{-1}$.*

Hence we have fully classified the multiplicative preservers of the C-numerical range and radius where C is a scalar matrix.

To complete our classification, we consider the case where C is not a scalar matrix. In order to do this, we first consider the multiplicative preservers of the C-numerical radius, which are classified in Theorem 3.2 of [8] as follows:

Theorem 5. *Given $C \in M_n(\mathbb{C})$ where $C \neq \mu I$ for any $\mu \in \mathbb{C}$, a multiplicative map $\phi : \mathcal{H} \to M_n(\mathbb{C})$ preserves the C-numerical radius for all $A \in \mathcal{H}$ if and only if there is a unitary $U \in SL_n(\mathbb{C})$ and a map $f : \mathbb{C}^* \to S^1$ where $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$ such that one of the following conditions hold true:*

1. *ϕ has the form $A \mapsto f(\det(A))UAU^*$, or*

2. *there exists $\mu \in S^1$, $V \in \mathcal{U}_n(\mathbb{C})$ such that $\mu\overline{C} = VCV^*$ and ϕ has the form $A \mapsto f(\det(A))U\overline{A}U^*$.*

To address the multiplicative preservers of the C-numerical range for a non-scalar matrix C, we will first define some terms. A *block matrix* is a square matrix that can be partitioned into smaller square matrices. As an example, we consider the following matrix:

$$X = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

We split this 6×6 matrix X into a 3×3 block matrix of 2×2 matrices. Labeling each sub-matrix X_{ij} where i gives the row in blocked form and j the column, we write $X = (X_{ij})$. This particular matrix X said to be in *block shift form*. A block matrix is in block shift form if all the diagonal blocks are square matrices, and $X_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ whenever $j \neq i + 1$. We can see X is in block shift form as each of the diagonal blocks are square matrices, and all of the $X_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ except $X_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $X_{23} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Now we can give our final classification for multiplicative preservers of the C-numerical range, which is from Theorem 4.2 in [8]:

Theorem 6. *Let $C \in M_n$ be a non-scalar matrix. A multiplicative map $\phi : \mathcal{H} \to M_n$ satisfies $W_C(\phi(A)) = W_C(A)$ if and only if there is a unitary matrix $U \in SL_n(\mathbb{C})$ and a map $f : \mathbb{C}^* \to S^1$ where $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$ such that one of the following holds true:*

1. *ϕ has the form $A \mapsto UAU^*$,*
2. *there exists a unitary matrix V and a matrix X in block shift form such that $C = VXV^*$ and ϕ has the form $A \mapsto f(\det(A))UAU^*$, or*
3. *there exists a unitary matrix V and a matrix X in block shift form such that $C = VXV^*$, there exists a unitary matrix W such that $\mu\overline{C} = WCW^*$ for some $\mu \in S^1$, and ϕ has the form $A \mapsto f(\det(A))U\overline{A}U^*$.*

Therefore we now have surveyed the complete classification of multiplicative maps that preserve the C-numerical range and radius.

3 Variations on multiplicative preservers

In a talk on numerical range and radius preserving maps, Chi-Kwong Li of the College of William and Mary, a prolific researcher in the field, proposed considering maps that are homomorphisms with respect to other binary operations such as $A * B = AB^{-1}$, $A \circ B = AB^*$, or $A \bullet B = AB - BA$. Using this as inspiration for research, our goal was to take these new operations and find analogues to the theorems which classify the multiplicative preservers of C-numerical range and radius.

We will call an operation $*$-multiplicative if it is a homomorphism with respect to $*$. In other words $\phi : (GL_n(\mathbb{C}), *) \to (M_n(\mathbb{C}), *)$ is $*$-multiplicative if and only if $\phi(A * B) = \phi(A) * \phi(B)$ for all $A, B \in GL_n(\mathbb{C})$. Given any other binary operations, such as $\circ$, then we would say a mapping is $\circ$-multiplicative if it is a homomorphism with respect to $\circ$, etc.

We began our research looking at $*$ defined by $A * B = AB^{-1}$. We want to begin classifying the $*$-multiplicative preservers of the C-numerical range and radius by computing specific examples to grasp the nature of this new binary operation.

3.1 $*$-multiplicative preserver examples

We consider the forms in Theorems 5 and 6 to determine whether they are also $*$-multiplicative preservers of the C-numerical range or radius. We started with the C-numerical radius, and thus tried the form *1* from Theorem 5. Let $U \in SL_n$ be unitary and $f : \mathbb{C}^* \to S^1$ be a map such that $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$. Define $\phi(A) = f(\det(A))UAU^*$. We note $f(1) = 1$ as $f(1) = f(1 \cdot 1) = f(1)f(1)$ which implies $f(1) = 1$.

To show ϕ is $*$-multiplicative, let $A, B \in \mathcal{H}$. Since all matrices in $\mathcal{H}$ are invertible, we know $\det(A)$ and $\det(B)$ are nonzero. Also recall $\det(B^{-1}) = \dfrac{1}{\det(B)} = \det(B)^{-1}$. We note $\phi(B^{-1}) = \phi(B)^{-1}$ as

$$1 = f(1) = f\left(\frac{\det(B)}{\det(B)}\right) = f\left(\frac{1}{\det(B)}\right) f(\det(B)) = f(\det(B^{-1}))f(\det(B)), \tag{3}$$

and thus since $0 \notin f(\mathbb{C}^*)$, we can divide the above equation by $f(\det(B))$ to conclude $f(\det(B))^{-1} = \dfrac{1}{f(\det(B))} = f(\det(B^{-1}))$. Hence,

$$\begin{aligned}\phi(B)^{-1} &= (f(\det(B))UBU^*)^{-1}\\ &= f(\det(B))^{-1}(UBU^*)^{-1}\\ &= f(\det(B))^{-1}UB^{-1}U^*\\ &= f(\det(B^{-1}))UB^{-1}U^*\\ &= \phi(B^{-1}).\end{aligned} \tag{4}$$

Recalling the determinant is multiplicative,

$$\begin{aligned}\phi(A * B) &= \phi(AB^{-1})\\ &= f(\det(AB^{-1}))U(AB^{-1})U^*\\ &= f(\det(A)\det(B^{-1}))(UAU^*)(UB^{-1}U^*)\\ &= f(\det(A))(UAU^*)f(\det(B^{-1}))(UB^{-1}U^*)\\ &= \phi(A)\phi(B^{-1})\\ &= \phi(A)\phi(B)^{-1}\\ &= \phi(A) * \phi(B).\end{aligned}$$

Hence ϕ is $*$-multiplicative.

Next we check to see if ϕ preserves the C-numerical range.

$$\begin{aligned} W_C(\phi(A)) &= \{\mathrm{tr}(CV\phi(A)V^*) : V \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{\mathrm{tr}(CVf(\det(A))UAU^*V^*) : V \in \mathcal{U}_n(\mathbb{C})\} \\ &= \{f(\det(A))\mathrm{tr}(CWAW^*) : W \in \mathcal{U}_n(\mathbb{C})\} \\ &= f(\det(A))W_C(A). \end{aligned}$$

If ϕ is restricted to all matrices of determinant 1, $W_C(\phi(A)) = W_C(A)$. Since this is not always the case, we conclude that in general, ϕ is not a $*$-multiplicative preserver of the C-numerical range. However if f is the constant function 1, then $\phi(A) = UAU^*$, and ϕ does preserve the numerical range. We will cover this later in the paper.

Though ϕ does not always preserve the numerical range, ϕ always preserves the C-numerical radius. If $y \in W_C(A)$, then $|f(\det(A))y| = |f(\det(A))| \cdot |y| = |y|$ since f maps into the unit circle. We then consider the C-numerical radius

$$\begin{aligned} w_C(\phi(A)) &= \max\{|z| : z \in W_C(\phi(A))\} \\ &= \max\{|z| : z \in f(\det(A))W_C(A)\} \\ &= \max\{|z| : z \in W_C(A)\} \\ &= w_C(A). \end{aligned}$$

Therefore ϕ is a $*$-multiplicative preserver of the C-numerical radius.

For one more example, consider part *2* of Theorem 5. Let $\phi(A) = f(\det(A))U\overline{A}U^*$ where U is a unitary matrix in SL_n and $f : \mathbb{C}^* \to S^1$ a map such that $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$. Suppose there exists $\mu \in S^1$ and $V \in \mathcal{U}_n(\mathbb{C})$ such that $\mu\overline{C} = VCV^*$. By Lemma 3.1 of [8], $w_C(A) = w_C(\overline{A})$.

We note

$$\begin{aligned} \phi(B)^{-1} &= \left(f(\det(B))U\overline{B}U^*\right)^{-1} \\ &= f(\det(B))^{-1}\left(U\overline{B}U^*\right)^{-1} \\ &= f(\det(B))^{-1}U\overline{B^{-1}}U^* \\ &= f(\det(B^{-1}))U\overline{B^{-1}}U^* \\ &= \phi(B^{-1}). \end{aligned}$$

Thus,

$$\begin{aligned}\phi(A * B) &= \phi(AB^{-1})\\ &= f(\det(AB^{-1}))U(\overline{AB^{-1}}U^*)\\ &= f(\det(A))(U\overline{A}U^*)f(\det(B^{-1}))U\overline{B^{-1}}U^*\\ &= \phi(A)\phi(B^{-1})\\ &= \phi(A)\phi(B)^{-1}\\ &= \phi(A) * \phi(B).\end{aligned}$$

Also, recalling f maps into S^1 so $|f(z)| = 1$ for all $z \in \mathbb{C}^*$,

$$\begin{aligned}w_C(\phi(A)) &= \max\{|z| : z \in \{\mathrm{tr}(CV\phi(A)V^* : V \in \mathcal{U}_n(\mathbb{C})\}\}\\ &= \max\{|z| : z \in f(\det(A))\{\mathrm{tr}(CVU\overline{A}U^*V^* : V \in \mathcal{U}_n(\mathbb{C})\}\}\\ &= \max\{|z| : z \in \{\mathrm{tr}(CW\overline{A}W^* : W \in \mathcal{U}_n(\mathbb{C})\}\}\\ &= w_C(\overline{A})\\ &= w_C(A).\end{aligned}$$

Hence ϕ is a $*$-multiplicative preserver of C-numerical radius.

3.2 Non-examples

If a map ϕ has any of the forms from Theorem 5, then ϕ is a $*$-multiplicative preserver of the C-numerical radius. Thus either $*$-multiplicative preservers of the C-numerical radius exactly coincide with multiplicative preservers of the C-numerical radius, or else there are maps of other forms which are also $*$-multiplicative preservers of the C-numerical radius.

To explore which option holds, we first checked if our unitary matrix from our previous form of ϕ had to be unitary, so we let S be a non-unitary matrix in $GL_n(\mathbb{C})$. We then defined ϕ by $\phi(A) = SAS^{-1}$. Our function ϕ is $*$-multiplicative as

$$\phi(A*B) = S(AB^{-1})S^{-1} = (SAS^{-1})(SB^{-1}S^{-1}) = (SAS^{-1})(SBS^{-1})^{-1} = \phi(A)*\phi(B).$$

For the C-numerical range, consider

$$W_C(\phi(A)) = \{\mathrm{tr}(CVSAS^{-1}V^* : V \in \mathcal{U}_n(\mathbb{C})\}.$$

If S were unitary, this gives us exactly $W_C(A)$ as the product of a unitary matrix by the set of all unitary matrices is $\mathcal{U}_n(\mathbb{C})$. Since S is not unitary, we want to find out more about the product of $S\mathcal{U}_n(\mathbb{C}) = \{SU : U \in \mathcal{U}_n(\mathbb{C})\}$. If $US = VS$ for some $U, V \in \mathcal{U}_n(\mathbb{C})$, we right-multiply by S^{-1} to find $U = V$. Thus the number of elements in $\mathcal{U}_n(\mathbb{C})$ is equal to the number of elements in $S\mathcal{U}_n(\mathbb{C})$. However, not every element in $S\mathcal{U}_n(\mathbb{C})$ is unitary. As a counterexample, let

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix},$$

and let $U = I$ where I is the $n \times n$ identity matrix. Then, $SU = SI = S$, but $S^* = S$ and

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n} \end{bmatrix}.$$

Thus we have a matrix $SI \in S\mathcal{U}_n(\mathbb{C})$ that is not unitary as $S^* \neq S^{-1}$. Thus in general ϕ does not necessarily preserve the C-numerical range or radius since $\{CVSAS^{-1}V^* : V \in \mathcal{U}_n(\mathbb{C})\}$ is not necessarily the same set as $\{CUAU^* : U \in \mathcal{U}_n(\mathbb{C})\}$. We note however that $\{CVSAS^{-1}V^* : V \in \mathcal{U}_n(\mathbb{C})\} \neq \{CUAU^* : U \in \mathcal{U}_n(\mathbb{C})\}$ does not necessarily imply $\{\text{tr}(CVSAS^{-1}V^*) : V \in \mathcal{U}_n(\mathbb{C})\} \neq \{\text{tr}(CUAU^*) : U \in \mathcal{U}_n(\mathbb{C})\}$. The point is that we are not guaranteed that a map of the form $A \mapsto SAS^{-1}$ will always preserve the numerical range or radius.

So we have a $*$-multiplicative function and conjecture that this map does not always preserve the C-numerical radius. Hence our goal is to find a specific S, A, and C for which ϕ does not preserve the C-numerical radius. We will not go through all the examples, but we will give a few examples to help explain the non-triviality of this quest as well as the thought process involved in searching for a counterexample.

We first show that both S and A cannot be diagonal. Otherwise,

$$\begin{aligned} \phi(A) &= SAS^{-1} \\ &= \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{bmatrix} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} \frac{1}{s_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s_n} \end{bmatrix} \\ &= \begin{bmatrix} s_1 a_1 \frac{1}{s_1} & 0 & \cdots & 0 \\ 0 & s_2 a_2 \frac{1}{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n a_n \frac{1}{s_n} \end{bmatrix} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} = A. \end{aligned}$$

We also have that regardless of A and C, S cannot be a scalar matrix as $S = \mu I$ implies $S^{-1} = \frac{1}{\mu}I$, thus $\phi(A) = SAS^{-1} = \mu I A \frac{1}{\mu} I = A$.

Therefore, to find a ϕ of the form $A \mapsto SAS^{-1}$ that will not preserve the C-numerical range of some matrix A, we cannot have both S and A be diagonal, nor can S be a scalar matrix. From this point on, all the examples considered involve only 2×2 matrices as we have an explicit form for the general unitary matrices.

Let C be diagonal, S be diagonal, and A be an upper triangular matrix. So, $S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} \frac{1}{s_1} & 0 \\ 0 & \frac{1}{s_2} \end{bmatrix}$, $A = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}$, and $C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ for $s_i, a_i, c_i \in \mathbb{C}$. We compute

$$\phi(A) = SAS^{-1} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \begin{bmatrix} \frac{1}{s_1} & 0 \\ 0 & \frac{1}{s_2} \end{bmatrix} = \begin{bmatrix} s_1 a_1 & s_1 a_2 \\ 0 & s_2 a_3 \end{bmatrix} \begin{bmatrix} \frac{1}{s_1} & 0 \\ 0 & \frac{1}{s_2} \end{bmatrix} = \begin{bmatrix} a_1 & \frac{s_1}{s_2} a_2 \\ 0 & a_3 \end{bmatrix}.$$

Thus,

$$\begin{aligned}
&W_C(\phi(A)) \\
&= \{\mathrm{tr}(CU\phi(A)U^*) : U \in \mathcal{U}_2(\mathbb{C})\} \\
&= \left\{ \mathrm{tr}\left(\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ \overline{u_2} & -\overline{u_1} \end{bmatrix} \begin{bmatrix} a_1 & \frac{s_1}{s_2} a_2 \\ 0 & a_3 \end{bmatrix} \begin{bmatrix} \overline{u_1} & u_2 \\ \overline{u_2} & -u_1 \end{bmatrix} \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ \mathrm{tr}\left(\begin{bmatrix} c_1 u_1 & c_1 u_2 \\ c_2 \overline{u_2} & -c_2 \overline{u_1} \end{bmatrix} \begin{bmatrix} a_1 \overline{u_1} + \frac{s_1}{s_2} a_2 \overline{u_2} & a_1 u_2 - \frac{s_1}{s_2} a_2 u_1 \\ a_3 \overline{u_2} & -a_3 u_1 \end{bmatrix} \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ c_1 a_1 |u_1|^2 + \frac{s_1}{s_2} c_1 a_2 u_1 \overline{u_2} + c_1 a_3 |u_2|^2 + c_2 a_1 |u_2|^2 \right. \\
&\qquad \left. - \frac{s_1}{s_2} c_2 a_2 u_1 \overline{u_2} + c_2 a_3 |u_1|^2 : |u_1|^2 + |u_2|^2 = 1 \right\}.
\end{aligned}$$

Now that we have this form, we then will try a few values to gain some more intuition. Let $C = S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$,

$$\begin{aligned}
&W_C(\phi(A)) \\
&= \left\{ a_1 |u_1|^2 + \frac{1}{2} a_2 u_1 \overline{u_2} + a_3 |u_2|^2 + 2a_1 |u_2|^2 - a_2 u_1 \overline{u_2} + 2a_3 |u_1|^2 : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ 2a_1 + a_3 + |u_1|^2 (a_3 - a_1) - \frac{1}{2} a_2 u_1 \overline{u_2} : |u_1|^2 + |u_2|^2 = 1 \right\}.
\end{aligned}$$

Also

$$
\begin{aligned}
&W_C(A) \\
&= \{\mathrm{tr}(CUAU^*) : U \in \mathcal{U}_2(\mathbb{C})\} \\
&= \left\{ \mathrm{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ \overline{u_2} & -\overline{u_1} \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \begin{bmatrix} \overline{u_1} & u_2 \\ \overline{u_2} & -u_1 \end{bmatrix} \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ \mathrm{tr}\left(\begin{bmatrix} u_1 & u_2 \\ 2\overline{u_2} & -2\overline{u_1} \end{bmatrix} \begin{bmatrix} a_1\overline{u_1} + \frac{s_1}{s_2}a_2\overline{u_2} & a_1u_2 - \frac{s_1}{s_2}a_2u_1 \\ a_3\overline{u_2} & -a_3u_1 \end{bmatrix} \right) : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ a_1|u_1|^2 + \frac{1}{2}a_2u_1\overline{u_2} + a_3|u_2|^2 + 2a_1|u_2|^2 - \frac{1}{2}a_2u_1\overline{u_2} + 2a_3|u_1|^2 : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \left\{ 2a_1 + a_3 + |u_1|^2(a_3 - a_1) : |u_1|^2 + |u_2|^2 = 1 \right\}.
\end{aligned}
$$

Remember our goal is to find a matrix A such that ϕ does not preserve the C-numerical radius. Given any values for a_1, a_2, and a_3, we can explicitly find the C-numerical radius. Thus as an example consider $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$. Then

$$
\begin{aligned}
w_C(A) &= \max\{|2a_1 + a_3 + |u_1|^2(a_3 - a_1)| : 0 \leq |u_1|^2 \leq 1\} \\
&= \max\{|5 + 2|u_1|^2| : 0 \leq |u_1|^2 \leq 1\} \\
&= 7.
\end{aligned}
$$

Now we have $w_C(A) = 7$, and we need to compare it to $w_C(\phi(A))$. We cannot directly compute $w_C(\phi(A))$. However, for any u_1' and u_2',

$$
\begin{aligned}
w_C(\phi(A)) &= \max\left\{ \left| 2a_1 + a_3 + |u_1|^2(a_3 - a_1) - \frac{1}{2}a_2u_1\overline{u_2} \right| : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&= \max\left\{ \left| 5 + 2|u_1|^2 - u_1\overline{u_2} \right| : |u_1|^2 + |u_2|^2 = 1 \right\} \\
&\geq \left| 5 + 2|u_1'|^2 - u_1'\overline{u_2'} \right|.
\end{aligned}
$$

Thus our goal is to find u_1' and u_2' such that $\left|5 + 2|u_1'|^2 - u_1'\overline{u_2'}\right| > 7 = w_C(A)$, which would give us an example where ϕ does not preserve the C-numerical radius. One strategy used was guess and check, which was unsuccessful. Another strategy is writing out $u_1 = x_1 + iy_1$ and $u_2 = x_2 + iy_2$. From this information, we can write $w_C(\phi(A))$ explicitly in terms of the real variables x_1, x_2, y_1, and y_2. Thus we are trying to maximize a value in terms of 4 real variables subject to the constraint that $x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1$. However the standard method of applying Lagrange multipliers to this optimization problem was not successful in finding a u_1' and u_2' that gave us our desired results.

Therefore we concluded that we are unable to find a counterexample with these particular values. Other values for $s_1, s_2, c_1, c_2, a_1, a_2$, and a_3 were tried, again to no avail as well as considering S as an upper triangular matrix using the same methods as above. Given the lack of success, we believe the best approach to finding a counterexample would be to use a computer.

Though it would be useful to have a counterexample, this is not a necessary step to find the general form of all $*$-multiplicative preservers of the C-numerical radius. We instead conjecture that the forms would closely match those found for standard multiplicative preservers, and in the next section we look at the converse of Theorem 4 for a specific C matrix.

3.3 Converse for $C = \mu I$

The next strategy is to prove the converse of our conjecture for a specific case where $C = \mu I$ with $\mu \in \mathbb{C}^*$. In other words, we want to start with a $*$-multiplicative map that preserves the C-numerical radius and make deductions about the form of ϕ. So we assume ϕ is a $*$-multiplicative map that preserves the C-numerical range. From equations (2) and (1), $W_C(\phi(A)) = \mu\text{tr}(\phi(A))$ and $W_C(A) = \mu\text{tr}(A)$. Since we assume ϕ preserves the C-numerical radius, we have $|\text{tr}(A)| = |\text{tr}(\phi(A))|$. Thus our problem reduces to finding an analogue of Theorem 4. This however is not trivial. As we started to look into the proofs of Theorem 4 in [3], we discovered they are all dependent on the fact that a multiplicative map gives a group homomorphism.

This observation led us to question whether or not $*$ gives a group structure on $M_n(\mathbb{C})$. The resulting answer is that $*$ does not give a group structure on $M_n(\mathbb{C})$ as $*$ is not associative. To see this, let $A, B, C \in M_n(\mathbb{C})$. Then

$$\begin{aligned} A * (B * C) &= A * (BC^{-1}) = A(BC^{-1})^{-1} = ACB^{-1} \\ &\neq AB^{-1}C^{-1} = (AB^{-1}) * C = (A * B) * C. \end{aligned}$$

Hence $*$ is not associative and $(M_n(\mathbb{C}), *)$ is not a group. Since this argument did not depend on the determinant of the matrices, we have $(GL_n(\mathbb{C}), *)$ and $(SL_n(\mathbb{C}), *)$ are also not groups. The lack of a group structure ended our ability to use any of the previously developed background, which led us to question the binary operation $*$ more extensively.

3.4 Back to Basics

In further exploring the basic structure of our binary operation $*$ we were able to conclude that any given map that is $*$-multiplicative is also multiplicative in the standard sense. To see this, suppose $\phi : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is a map such that $\phi(A * B) = \phi(A) * \phi(B)$.

Note $\phi(I) = \phi(I * I) = \phi(I) * \phi(I) = \phi(I)\phi(I)^{-1}$. Since ϕ must map into $GL_n(\mathbb{C})$, we can left-multiply by $\phi(I)^{-1}$ to get $I = \phi(I)^{-1}$. Thus $I = \phi(I)$.

Now let $C \in GL_n(\mathbb{C})$. Then $\phi(C^{-1}) = \phi(I*C) = \phi(I)*\phi(C) = \phi(I)\phi(C)^{-1} = \phi(C)^{-1}$.

Finally let $A, B \in GL_n(\mathbb{C})$. Then

$$\phi(AB) = \phi(A * B^{-1}) = \phi(A) * \phi(B^{-1}) = \phi(A) * \phi(B)^{-1} = \phi(A)\phi(B),$$

and thus ϕ is multiplicative.

Using this argument, every map that is a $*$-multiplicative preserver of the C-numerical range [resp. radius] has the forms in Theorems 3, 4, 5, and 6. We can also verify that each of those forms is $*$-multiplicative.

As an example, we consider Theorem 4 where $n = 2$ and $\mathcal{H} = SL_2(\mathbb{C})$ and ϕ has the form $A \mapsto S(A^{-1})^T S^{-1}$. If ϕ has this form, it is $*$-multiplicative as

$$\begin{aligned}\phi(A * B) &= S((AB^{-1})^{-1})^T S^{-1} \\ &= S(BA^{-1})^T S^{-1} \\ &= S(A^{-1})^T B^T S^{-1} \\ &= S(A^{-1})^T S^{-1}(S(B^T)S^{-1}) \\ &= \phi(A)(S(B^{-1})^T S^{-1})^{-1} \\ &= \phi(A)\phi(B)^{-1} = \phi(A) * \phi(B).\end{aligned}$$

To see that form *3* of Theorem 4 is indeed $*$-multiplicative, we let ϕ have the form $A \mapsto f(\det(A))SAS^{-1}$. Since f and the determinant are both multiplicative functions,

$$f(\det(AB^{-1})) = f(\det(A)\det(B^{-1})) = f(\det(A))f\left(\frac{1}{\det(B)}\right) = f(\det(A))f(\det(B))^{-1}.$$

Thus, by 3 and 4, we have

$$\begin{aligned}\phi(A * B) &= f(\det(AB^{-1}))SAB^{-1}S^{-1} \\ &= f(\det(A))(f(\det(B)))^{-1}SA(S^{-1}S)B^{-1}S^{-1} \\ &= (f(\det(A))SAS^{-1})(f(\det(B)))^{-1}SB^{-1}S^{-1} \\ &= \phi(A)\phi(B)^{-1} = \phi(A) * \phi(B).\end{aligned}$$

Using the same process as above on each of the forms, we can verify that all the forms that are multiplicative are also $*$-multiplicative. Thus we concluded that the set of $*$-multiplicative maps that preserve the C-numerical range and radius exactly match the set of multiplicative maps that preserver the C-numerical range and radius. We now state these as corollaries.

Corollary 7. *(Theorem 3). A multiplicative map $\phi : \mathcal{H} \to M_n(\mathbb{C})$ satisfies $W_C(\phi(A)) = W_C(A)$ for C a scalar matrix and for all $A \in \mathcal{H}$ if and only if there is an $S \in SL_n(\mathbb{C})$ such that*

1. *ϕ has the form $A \mapsto SAS^{-1}$, or*
2. *if $n = 2$, and $\mathcal{H} = SL_2(\mathbb{C})$, ϕ has the form $A \mapsto S(A^{-1})^T S^{-1}$.*

Corollary 8. *(Theorem 4). A $*$-multiplicative map $\phi : \mathcal{H} \to M_n(\mathbb{C})$ satisfies $w_C(\phi(A)) = w_C(A)$ for C a scalar matrix and for all $A \in \mathcal{H}$ if and only if there is an $S \in SL_n(\mathbb{C})$ such that*

1. *ϕ has the form $A \mapsto SAS^{-1}$ or $A \mapsto S\overline{A}S^{-1}$,*
2. *if $n = 2$, and $\mathcal{H} = SL_2(\mathbb{C})$, ϕ has the form $A \mapsto S(A^{-1})^T S^{-1}$, or*
3. *if $\mathcal{H} = GL_n(\mathbb{C})$, for a map $f : \mathbb{C}^* \to S^1$ where $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$, ϕ has the form $A \mapsto f(\det(A))SAS^{-1}$ or $A \mapsto f(\det(A))S\overline{A}S^{-1}$.*

Corollary 9. *(Theorem 5). Given $C \in M_n(\mathbb{C})$ where $C \neq \mu I$ for any $\mu \in \mathbb{C}$, a multiplicative map $\phi : \mathcal{H} \to M_n(\mathbb{C})$ preserves the C-numerical radius for all $A \in \mathcal{H}$ if and only if there is a unitary $U \in SL_n(\mathbb{C})$ and a map $f : \mathbb{C}^* \to S^1$ where $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$ such that one of the following conditions hold true:*

1. *ϕ has the form $A \mapsto f(\det(A))UAU^*$, or*
2. *there exists $\mu \in S^1$, $V \in \mathcal{U}_n(\mathbb{C})$ such that $\mu\overline{C} = VCV^*$ and ϕ has the form $A \mapsto f(\det(A))U\overline{A}U^*$.*

Corollary 10. *(Theorem 6). Let $C \in M_n$ be a non-scalar matrix. A multiplicative map $\phi : \mathcal{H} \to M_n$ satisfies $W_C(\phi(A)) = W_C(A)$ if and only if there is a unitary matrix $U \in SL_n(\mathbb{C})$ and a map $f : \mathbb{C}^* \to S^1$ where $f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^*$ such that one of the following holds true:*

1. *ϕ has the form $A \mapsto UAU^*$,*
2. *there exists a unitary matrix V and a matrix X in block shift form such that $C = VXV^*$ and ϕ has the form $A \mapsto f(\det(A))UAU^*$, or*
3. *there exists a unitary matrix V and a matrix X in block shift form such that $C = VXV^*$, there exists a unitary matrix W such that $\mu\overline{C} = WCW^*$ for some $\mu \in S^1$, and ϕ has the form $A \mapsto f(\det(A))U\overline{A}U^*$.*

Recall we originally wanted to look at other binary operations and have only considered the binary operation $*$. This is the only binary operation we tried for which ϕ was both $*$-multiplicative and multiplicative. The other binary operations tried were not multiplicative, nor did they give a group structure. With a lack of group structure, we cannot use any of the existing theorems about the structure of multiplicative maps, and lacking such structure, any analogues of these theorems would have to be proven using a radically different approach. The following is a list of binary operations we tried that are not multiplicative, nor do they give a group structure as they are not associative:

- $A \circ B = AB^*$,
- $A \circ B = AB - BA$,
- $A \circ B = ABA^{-1}$,
- $A \circ B = BAB^{-1}$,
- Given a fixed $X \in GL_n$, $A \circ B = XABX^{-1}$, and
- Given $k \in \mathbb{Z} \setminus \{0, 1\}$, $A \circ B = (AB)^k$.

We also considered the mapping $A \bullet B = A$. This mapping does trivially give a group structure on $M_n(\mathbb{C})$, but any operation on M_n will be $\bullet$-multiplicative as $\phi(A \bullet B) = \phi(A) = \phi(A) \bullet \phi(B)$. We also considered the binary operation given by element-wise multiplication but were unable to explore this question thoroughly due to time constraints. We leave this as an open problem.

4 Conclusion

In order to reach conclusions about variations on multiplicative preservers of the C-numerical range or radius, it is essential to have a binary operation which gives a group structure if we are to build from existing results. Very few of these binary operations exist to our knowledge. If they do, like $*$, they are simply multiplicative. As further research, one could find a binary operation that gives a group structure on the matrices or pursue element-wise multiplication. It is still an open question whether or not some theorems can be built using the non-associative binary operations; however we are skeptical as to the benefits of pursuing this particular option as non-associative binary operations give very little structure to a set.

References

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[6] Chi-Kwong Li. C-numerical ranges and C-numerical radii. *Linear and Multilinear Algebra*, 37:51–82, 1994.

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5 Appendix A: Literature Review

The following notes give a survey of papers read and books referenced during this project specifically related to the research presented in this paper. They are cited in order from a recommended start down through the most specialized papers to help give an idea of starting points for anyone interested in starting research in this field.

[9] This survey is useful for first gaining familiarity working with the numerical range. It begins by defining the classical numerical range, giving definitions, properties, and proving major theorems related to the convexity and geometry of the numerical range. It also presents open problems, giving the reader a thorough understanding of the field of research.

[1] This is a reference for the linear algebra used in research with the C-numerical range. Though it is not fully matrix based, the deeper theory that isn't found in a standard undergraduate linear algebra book is very helpful.

[4] The extent to which Hungerford thoroughly treats algebra in the most general sense possible makes this text a great reference. In particular when looking at the binary operations that don't give a group, he defines a homomorphism in terms of semigroups, which just require closure and associativity. Because Hungerford gives the most generalized statements of theorems possible, one can easily reference exactly what is needed for standard theorems to hold.

[6] This source starts with the classical numerical range and proceeds to develop properties analogous to those found in [9]. It is not as accessible to read as [9], but in terms of the C-numerical range and radius it is a very comprehensive start as well as a useful reference.

[7] This survey paper considers linear preservers of various numerical ranges and radii. In particular it gives general techniques used for working with preserver problems related to numerical ranges and radii. It then proceeds to take a variation on the numerical range, connect it to other variations, and provide references, survey the known theorems related to preservers, and give ideas for further research.

[2] This paper gives an overview for multiplicative preservers on semigroups of matrices. It gives a good idea of techniques and questions asked when working with preserver problems as well as the standard conjugation form for multiplicative maps that preserve various functions.

[3] This paper takes a general group approach to build up theorems for multiplicative maps on $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ that preserve properties of matrices, including a treatment of the trace as well as numerical ranges.

[8] This paper gives the full classification of the forms of multiplicative preservers of C-numerical ranges and radii. It presents three main theorems with in-depth work through the proofs. Thus it develops a strong and deep understanding of multiplicative preservers of C-numerical ranges and radii.

6 Appendix B: Reflection Section

As a student who has completed an honors project, I write this section to provide an advice-based anecdotal reflection on the mathematical research process, as well as the process in the context of being a student.

Dear student considering an honors project in mathematics,

My first advice is to not expect too much. Results are not guaranteed and finding a question takes a lot of time. The nature of research is a lot of failure. I started out in math research seminar spring of my junior year working on the classical numerical range of matrices over finite fields. I did a lot of research and examples with this topic, but something was not quite right, and it turns out I had been using a standard inner product formula over the integers which does not give an inner product over finite fields. Even when I did have an inner product, I still did not have a norm which is necessary for the classical numerical range. In the last week of the semester when I finally had this realization, my honors proposal was completely changed, and I had to start from square one again. After picking a new topic, I spent my senior year working on multiplicative preservers and I was using theorems that required a group homomorphism, but I didn't have a group with my new binary operations. Of course I discovered this three weeks before my final paper was due. Research is more about the process. I enjoy the process, and regardless of having results, I still learned a lot of information through this process.

Next, I learned how much time research takes. I had been in two mathematics-based Research Experiences for Undergraduates (REU's), but I did not fully appreciate the freedom of having nothing to do but research for weeks at a time. Balancing a schedule of commitments isn't easy and requires putting research as a top priority. I was applying for graduate schools as well as visiting graduate schools, and with more time I might have been able to explore more topics and go above and beyond the standard requirements.

I am thankful for this experience, and the aspect of research I experienced in this honors project was unlike any other research experience I had before in that I did not have a well-defined question to research. I got to get my hands messy, feel lost, and have no idea what I was even asking at times. So as you decide whether or not to pursue an honors project in mathematics, remember the time line is not necessarily set up for an abundance of original results given the nature of research in this field. So go in with an open mind, looking to learn, ask questions, and enjoy the research process of failures, successes, and even the nothings in between!

Best of luck in your honors project!

Sammy Fairchild